1 The Riemann-Roch Theorem

1.1 Introduction

Let $\mathcal{L}$ be a holomorphic line bundle over a compact Riemann surface $X$. We define $\bar{\partial} : \mathcal{C}^\infty(X, \mathcal{L}) \to \mathcal{C}^\infty(X, \mathcal{L} \otimes \Lambda^{0,1})$ by sending $s \mapsto \bar{\partial}z s$, which is well-defined as the transition functions $t_{\mu\nu}$ are holomorphic. We impose a metric $h$ on $\mathcal{L}$ and define $\deg(\mathcal{L}) = \int_X \bar{\partial}z \bar{\partial}z \log h = \deg(s)$ for any meromorphic section $s$ of $\mathcal{L}$. This quantity is manifestly independent of metric or section. We also impose a metric $g$ on $\Lambda^{0,1}$. Let $H^0(X, \mathcal{L})$ denote the space of holomorphic sections of $\mathcal{L}$ and let $h^0(\mathcal{L})$ denote the dimension of this space. The goal of this document is to provide a clear, comprehensible, and un-cumbersome proof of the following theorem:

**Theorem 1.1.1.** Let $\mathcal{L}$ be a holomorphic line bundle over a compact Riemann surface $X$. Then

$$h^0(\mathcal{L}) - h^0(\mathcal{L}^{-1} \otimes \Lambda^{1,0}) = \deg \mathcal{L} + 1 - g$$

where $g$ is the topological genus of $X$.

The strategy of our proof is as follows:

1. Prove that the index of a Fredholm operator is locally constant.
2. Define and describe the Sobolev spaces $W^k(X, \mathcal{L})$.
3. Prove that the $\bar{\partial}$ operator is Fredholm on the Sobolev spaces $W^k(X, \mathcal{L})$.
4. Show that the $\mathcal{C}^\infty$ cohomology is equal to the Sobolev space cohomologies.
5. Prove that the index of $\bar{\partial}$ depends only on the genus of $X$ and the degree of $\mathcal{L}$.
6. Explicitly compute the index of $\bar{\partial}$ for a point bundle on a hyperelliptic curve.

1.2 Fredholm Operators

Let $T$ be a bounded linear operator of Banach spaces (normed complete vector spaces over $\mathbb{C}$).

**Definition 1.2.1.** We say $T$ is Fredholm if the following three conditions hold:

1. $\dim \ker T$ is finite.
2. $\im T$ is closed.
3. $\dim \coker T$ is finite.

We call an operator semi-Fredholm if (2) and (1) or (3) holds. If $T$ is semi-Fredholm, we define the index

$$\text{ind}(T) := \dim \ker T - \dim \coker T,$$

in which case the index takes values in $\mathbb{Z} \cup \{\pm \infty\}$.

**Definition 1.2.2.** We say that $K$ is a compact operator if the image of the unit ball $B_1$ is sequentially compact.

**Theorem 1.2.3.** Let $K$ be a compact operator. Then $I - K$ is Fredholm with index 0.

**Proof.** If $\dim \ker (I - K)$ is infinite then pick an infinite set $\{v_i\}$ of it such that $||v_i|| \leq 1$ and

$$d(v_n, \text{span}(v_1, \ldots, v_{n-1})) > \frac{1}{2}.$$

Then, $v_i \in K(B_1)$, but $\{v_i\}$ is not sequentially compact. This is a contradiction, so $\dim \ker (I - K)$ is finite.

Now, suppose $\{v_i - Kv_i\} \to w$ and $v_i \in B_1$. Then, since $K$ is compact, a subsequence $Kv_i \to v$, hence $v_i \to w + v$. Now note $(I - K)(w + v) = w$ by continuity, implying $\im(B_1)$ is closed under $I - K$. Thus, its linear span, the whole image, is closed. Finally, by semi-Fredholm stability, proven below, $I - tK$ is a continuous family of semi-Fredholm operators, hence $\text{ind}(I - K) = \text{ind}(I) = 0$. \qed
Theorem 1.2.4 (semi-Fredholm stability). The index on the space of semi-Fredholm operators is locally constant in the operator norm topology.

Proof. Suppose \( T : X \to Y \) is semi-Fredholm. Decompose spaces as \( X = C \oplus \ker T \) and \( Y = \text{im} T \oplus D \). Then, we may represent the operator \( T \) as
\[
\begin{pmatrix}
T' & 0 \\
0 & 0
\end{pmatrix}
\]
with respect to this decomposition. Note that \( T' \) is an isomorphism from \( C \to \text{im} T \). Write
\[
p = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
\]
We may assume that \( T' + a \) is invertible by choosing \( ||p|| \) small enough. We claim that \( \text{ind}(T) = \text{ind}(T + p) \).

Define isomorphisms \( G : X \to X \) and \( H : Y \to Y \) by
\[
G = \begin{pmatrix}
I & -(T' + a)^{-1} b \\
0 & I
\end{pmatrix} \\
H = \begin{pmatrix}
I & 0 \\
-c(T' + a)^{-1} & I
\end{pmatrix}.
\]
Because \( G \) and \( H \) are isomorphisms
\[
\text{ind}(T + p) = \text{ind}(H(T + p)G) = \begin{pmatrix}
T' + a & 0 \\
0 & -c(T' + a)^{-1} b + d
\end{pmatrix}.
\]
Thus, \( \text{ind}(T + p) = \text{ind}(-c(T' + a)^{-1} b + d) = \text{ind}(T) \) by the rank-nullity theorem, which states that the index is the difference of dimensions of the source and target, if either space is finite dimensional.

Definition 1.2.5. Suppose that \( T : X \to Y \) is bounded. Then, a pseudoinverse is an operator \( P : Y \to X \) such that \( TP - I_Y \) and \( PT - I_X \) are compact. By the Theorem 1.2.3, existence of a pseudoinverse clearly implies Fredholmicity.

1.3 Sobolev Spaces

The intuition behind Sobolev spaces is that \( C^\infty(X, \mathcal{L}) \) is not complete with respect to obvious norms on this space. Thus we need to complete \( C^\infty(X, \mathcal{L}) \) to apply the theory of Fredholm operators to it. We complete with respect to different norms \( ||\cdot||_k \) to get the space \( W^k(X, \mathcal{L}) \). Then, we show
\[
\bigcap_k W^k(X, \mathcal{L}) = C^\infty(X, \mathcal{L})
\]
allowing us to deduce properties about the \( C^\infty \) space from properties of the Sobolev spaces.

An intuitive definition of \( ||s||_k \) is that we sum all of the \( L_2 \) norms of the derivatives of \( s \) up to order \( k \). While this definition is valid, it is not clear how to describe the completion \( W^k \) in this context, or how to define \( W^k \) for negative \( k \).

Let us first consider smooth functions on the torus \( C^\infty(T, \mathbb{C}) \), all of which have Fourier expansions
\[
\sum_{m,n} c_{mn} e^{2\pi i (nx + my)}.
\]
The \( L_2 \) completion of this space can simply be defined as
\[
\{ c_{mn} : \sum |c_{mn}|^2 < \infty \}.
\]
Furthermore, if we take the \( k \)th derivative of \( f(x, y) \) we get
\[
f^{(k)}(x, y) = \sum_{m,n} c_{mn} O((m^2 + n^2)^{k/2}) e^{2\pi i (mx + ny)},
\]
so the Sobolev space $W^k$ on the torus has a natural description as
\[
\{c_{mn} : \sum (1 + m^2 + n^2) + \cdots + (m^2 + n^2)^k |c_{mn}|^2 < \infty \} \quad \text{or} \quad \{c_{mn} : \sum (1 + m^2 + n^2)^k |c_{mn}|^2 < \infty \}.
\]
Thus, defining $W^k$ to be the space of $c_{mn}$ that satisfy that latter convergence condition provides us with a definition of $W^k$ that is easy to work with for all $k$. We call $k$ the regularity. Note that if $k \leq k'$ then $W^{k'} \subset W^k$.

To extend this definition to an arbitrary smooth line bundle on a Riemann surface, we use charts. Take a partition of unity $\{a_U\}$ subordinate to a finite set of trivializing charts $\{U\}$ of $\mathcal{L}$ over $X$, i.e. $\text{supp}(a_U) \subset U$. Then, we define the Sobolev norm
\[
||s||_k = \sum_U ||a_U s||_k
\]
by choosing a torus containing the chart $U$, and extending the function $a_U s$ by zero to an element of $C_\infty(T, \mathbb{C})$.

We have a linear map
\[
\mathcal{J} : W^k(X, \mathcal{L}) \to W^{k-1}(X, \mathcal{L} \otimes \Lambda^{0,1})
\]
induced by $\mathcal{J}$. The convergence condition on the Fourier series clearly implies that such a map exists, as
\[
\mathcal{J} : \{c_{mn}\} \mapsto \{(m - in)c_{mn}\}.
\]

1.4 Fredholmicity of $\mathcal{J}$

The goal of this section is to prove the following theorem:

**Theorem 1.4.1.** The following map is Fredholm:
\[
\mathcal{J} : W^k(X, \mathcal{L}) \to W^{k-1}(X, \mathcal{L} \otimes \Lambda^{0,1}).
\]

The method by which we prove this is by finding a pseudoinverse $P$ to $\mathcal{J}$. Let
\[
s \in W^{k-1}(X, \mathcal{L} \otimes \Lambda^{0,1}),
\]
and $\{b_U\}$ be bump functions such that $\text{supp}(b_U) \subset U$ and $b_U = 1$ on $\text{supp}(a_U)$. Write $b_U s = \{c_{mn}\}$ and define a bounded operator from $W^{k-1}(T, \mathbb{C}) \to W^k(T, \mathbb{C})$ by
\[
P_{\text{torus}} : \{c_{mn}\} \mapsto \{(m - in)^{-1}c_{mn}, mn \neq 00\} + c_{00} \overline{z}.
\]
Then $\mathcal{J}P_{\text{torus}} = I$ and $P_{\text{torus}}\mathcal{J} = I - c_{00}$. We then globally define our pseudoinverse
\[
P : s \mapsto \sum_U a_U P_{\text{torus}}(b_U s).
\]
Then,
\[
(\mathcal{J}P - I)s = \sum (\partial a_U) P_{\text{torus}}(b_U s) + a_U b_U s - a_U s
\]
\[
= \sum (\partial a_U) P_{\text{torus}}(b_U s)
\]
where the second equality holds by noting $b_U - 1 = 0$ in $\text{supp}(a_U)$. Define the operator
\[
R s = - \sum (\partial a_U) P_{\text{torus}}(b_U s)
\]
so that $\mathcal{J}P = I - R$ and similarly define $P\mathcal{J} = I - S$.

The operator $R$ is regularity increasing. This is clear from the definition of $R$, as $\partial a_U$ is a smooth $(0, 1)$-form, and $P_{\text{torus}}$ is regularity increasing. That is to say, $R$ is in fact a bounded operator from $W^{k-1}(X, \mathcal{L} \otimes \Lambda^{0,1}) \to W^k(X, \mathcal{L} \otimes \Lambda^{0,1})$. Similar arguments apply to $S$. Then, by the lemma below, $R$ and $S$ are compact because they can be written as the composition of a bounded operator with a compact operator. Hence $\mathcal{J}$ and $P$ are Fredholm.
Lemma 1.4.1 (Rellich’s Lemma). The inclusion \( W^k \to W^{k-1} \) is compact.

Proof. Suppose \( \{c_{mn}^i\} \in B_1 \) of \( W^k \). Then, we wish to show that in \( W^{k-1} \) there is a Cauchy subsequence. This is clear from splitting Fourier sum into two pieces

\[
\sum_{m^2+n^2 \leq C} c_{mn}^i e^{2\pi i (mx+ny)} + \sum_{m^2+n^2 > C} c_{mn}^i e^{2\pi i (mx+ny)}.
\]

We can find a sequence which is Cauchy in the first sum (as it contains only finitely many terms). Then, in \( W^{k-1} \) the remaining terms have norm bounded by \( O(C^{-1}) \). Taking a diagonal sequence as \( C \to \infty \) gives the desired Cauchy sequence.

\[\square\]

1.5 Relating \( C^\infty \) Cohomology to Sobolev Cohomology

Define \( H^i_k(X, \mathcal{L}) \) to be the cohomology of the complex

\[
0 \to W^k(X, \mathcal{L}) \to W^{k-1}(X, \mathcal{L} \otimes \Lambda^{0,1}) \to 0.
\]

Also, set \( H^i_\infty \) to be the cohomology of the the complex of smooth sections

\[
0 \to C^\infty(X, \mathcal{L}) \to C^\infty(X, \mathcal{L} \otimes \Lambda^{0,1}) \to 0.
\]

The goal of this section is to prove the following theorem:

Theorem 1.5.1. For any \( k, k' \in \mathbb{Z} \cup \{\infty\} \), the cohomology is isomorphic: \( H^i_k = H^i_{k'} \).

For finite \( k \), this cohomology is finite dimensional by the Fredholmicity of \( \mathcal{J} \). Note that if \( \mathcal{J}s = 0 \),

\[
P\mathcal{J}s = (I - S)s = 0
\]

implying that \( s = Ss \). As \( S \) is regularity increasing,

\[
s \in \bigcap_k W^k.
\]

Thus, there is a map \( H^0_k \to H^0_{k+1} \) inverse to the inclusion map, implying \( H^0_k = H^0_{k'} \) for all \( k, k' \in \mathbb{Z} \). If \( s \in \mathcal{J}(W^k) \) then \( s \in \mathcal{J}(W^{k-1}) \), so any closed form in \( W^k \) is closed in \( W^{k-1} \). Hence we have a map \( i : H^1_{k+1} \to H^1_k \).

Lemma 1.5.1. The map \( i : H^1_{k+1} \to H^1_k \) is an isomorphism.

Proof. First, let’s show injectivity. Suppose \( i(s) = 0 \). Then, \( s = \mathcal{J}t \) for some \( t \in W^k \). So, \( Ps = t - St \) which implies \( \mathcal{J}(St + Ps) = s \). Note that \( St + Ps \in W^{k+1} \), so \( s = 0 \) in \( H^1_{k+1} \).

Now, we check surjectivity. Suppose \( s \in W^{k-1} \). Then, \( \mathcal{J}P = I - R \), so \( s \) and \( Rs \) represent the same class in \( H^1_k \), implying \( i(Rs) = s \). \[\square\]

Even though \( i \) and \( R \) are isomorphisms, we don’t know if \( s, Rs, R^2s, \ldots \) converge to an element in \( \bigcap_k W^k \). To check this we utilize the special properties of \( H^1_1 \). In particular, \( W^0(X, \mathcal{L} \otimes \Lambda^{0,1}) \) has an inner product structure. So, we may define the cokernel of \( \mathcal{J} \) as an orthogonal complement:

\[
H^1_1 = \{ s : \langle s, \mathcal{J}t \rangle = 0 \text{ for all } t \in C^\infty(X, \mathcal{L}) \}
\]

where

\[
\langle f, g \rangle := \int_X f\overline{g} h.
\]

Note that for any \( u \in C^\infty(X, \mathcal{L} \otimes \Lambda^{0,1}) \) and \( s \in H^1_1 \) we have

\[
\langle s, u \rangle = \langle s, \mathcal{J}Pu + Ru \rangle = \langle s, Ru \rangle.
\]
Hence,
\[ \langle s, u \rangle = \langle s, R^k u \rangle \leq \|s\|_0 \|R^k u\|_0 \leq C\|u\|_{-k} \]
by boundedness of \( R \). Now we use that \((W^{-k})^* = W^k\). In particular, by choosing Fourier coefficients of \( u \) appropriately we can show \( s \in W^k(X, \mathcal{L} \otimes \Lambda^{0,1}) \). Hence
\[ s \in \bigcap_k W^k. \]

**Lemma 1.5.2 (Sobolev Embedding).**
\[ \bigcap_k W^k = C^\infty. \]

**Proof.** We prove by induction that \( W^k \subset C^{k-3} \). Note that if \( s \in W^{k+1} \), its derivative is in \( W^k \), and so in \( C^{k-3} \) by the inductive hypothesis. Hence, \( s \in C^{k-2} \). The base case is \( W^3 \subset C^0 \). We note that if
\[ \sum (1 + n^2 + m^2)^3 |c_{mn}|^2 < \infty \]
then \( |c_{mn}| \leq C(1 + n^2 + m^2)^{-3/2} \) for some \( C \). Thus, the Fourier series \( \{c_{mn}\} \) converges absolutely, to a continuous function. Thus, the lemma holds.

We now know that the cohomology classes have smooth representatives: \( H^1_\infty = H^1_1 = H^1_k \) and \( H^0_\infty = H^0_k \). Thus, we have proven Theorem 1.5.1.

### 1.6 Independence of complex structure

Suppose \( X \) and \( Y \) are two Riemann surfaces of equal genus. The complex structures on \( X \) and \( Y \) induce smooth structures. Since genus uniquely characterizes diffeomorphism classes of compact orientable smooth surfaces, there is a diffeomorphism between \( X \) and \( Y \) with these smooth structures. Pulling back the complex structure, we see that we may reduce to considering a single smooth surface \( X \) with two complex structures \( J \) and \( J' \) that are smoothly compatible. Suppose \( \mathcal{L} \) and \( \mathcal{L}' \) are holomorphic with respect to \( J \) and \( J' \) and are of equal degree.

First consider the case \( J = J' \). Then \( \mathcal{L} \) and \( \mathcal{L}' \) are smoothly isomorphic as complex line bundles by the first appendix. So viewing \( \overline{J}_{\mathcal{L}} \) and \( \overline{J}_{\mathcal{L}'} \) as operators from \( C^\infty(X, \mathcal{L}) \to C^\infty(X, \mathcal{L} \otimes \Lambda^{0,1}) \), we see that
\[ \overline{J}_{\mathcal{L}} s = \frac{ds}{dz} \quad \text{and} \quad \overline{J}_{\mathcal{L}'} s = b^{-1} \frac{d(bs)}{dz} = \frac{ds}{dz} + b^{-1} \frac{db}{dz}s \]
for some \( b \in C^\infty(X, \mathbb{C}^*) \). Write \( \omega = b^{-1} \frac{db}{dz} \). Then, \( \overline{J}_L + t\omega \) are a family of operators connecting \( \overline{J}_L \) and \( \overline{J}_{L'} \). They are Fredholm:
\[ P_L(\overline{J}_L + t\omega) = I - S + tP_L\omega \]
and \( tP_L\omega \) is regularity increasing. We have a similar computation for \( (\overline{J}_{L'} + t\omega)P_L \). Hence, by Fredholm stability \( \text{ind}(\overline{J}_L) = \text{ind}(\overline{J}_{L'}) \). So by the first appendix, we may assume that \( \mathcal{L} \) is a point bundle \( d[P] \).

By the second appendix, every almost complex structure on \( X \) gives rise to a complex structure. So, given any almost complex structure \( J \) on \( X \), we can define
\[ \overline{J}_{J,L} : C^\infty(X, \mathcal{L}) \to C^\infty(X, \mathcal{L} \otimes \Lambda^{0,1}_J) \]
where \( \Lambda^{0,1}_J \) is the \(-i\) eigenspace of \( J \). We claim the space of almost complex structures is connected. The fiber of \( JS \) is easy to describe: They are the elements of \( SL_2(\mathbb{R}) \) which square to \(-I\), implying they are of the form
\[ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \]
with \( b \neq 0 \). Since \( J \) and \(-J \) define isomorphic almost complex structures, we may assume \( b > 0 \). Then, the fiber is \( \mathbb{H} \), which is connected. Hence, the space of complex structures is connected.
We would like to compare $\mathcal{J}_{J,\mathcal{L}}$ and $\mathcal{J}_{J',\mathcal{L}'}$. It is sufficient to compare the bundles $d[P]$ with complex structures $J$ and $J'$. We note that if $J = J'$ in a neighborhood of $P$, then $C^\infty(X, \mathcal{L}) = C^\infty(X, \mathcal{L}')$ via the identity map, as the transition function for the bundle $d[P]$ is the same with either $J$ or $J'$. Thus,

$$\mathcal{J}_{J,\mathcal{L}} = \frac{ds}{d\pi_J} \quad \text{and} \quad \mathcal{J}_{J',\mathcal{L}'} = a_{J'} \frac{ds}{d\pi_{J'}},$$

where the transformation $a_{J'}$ sends $\Lambda_{J'}$ to $\Lambda_{J'}$. Note that $a_{J'}$ is the identity when $J' = J$ and varies smoothly with $J'$ because $\Lambda_{J'}$ varies smoothly with $J'$. So

$$(\mathcal{J}_{J,\mathcal{L}} - \mathcal{J}_{J',\mathcal{L}'})s = \frac{ds}{d\pi_J} - a_{J'} \frac{ds}{d\pi_{J'}} = (\pi_{J'}^0 - a_{J'} \pi_{J'}^0) \circ ds = \psi(\pi_{J'}^0 \circ ds)$$

for a real linear transformation $\psi \to 0$ as $J' \to J$. But then,

$$||\mathcal{J}_{J,\mathcal{L}} - \mathcal{J}_{J',\mathcal{L}'||} < \epsilon ||\mathcal{J}_{J,\mathcal{L}}|| < \epsilon.$$

Hence, by Fredholm stability $\text{ind}(\mathcal{J}_{J,\mathcal{L}}) = \text{ind}(\mathcal{J}_{J',\mathcal{L}'})$. Finally, we note that there is a sequence of perturbations of $J$ that fix some neighborhood of some point $P$, and transform it to any other $J'$.

**Theorem 1.6.1.** Suppose $\mathcal{L}$ and $\mathcal{L}'$ are holomorphic with respect to $J$ and $J'$. If $\text{deg}(\mathcal{L}) = \text{deg}(\mathcal{L}')$,

$$\text{ind}(\mathcal{J}_{J,\mathcal{L}}) = \text{ind}(\mathcal{J}_{J',\mathcal{L}'}).$$

### 1.7 The Hyperelliptic Curve

By the previous section it suffices to compute the index of $\partial$ for a point bundle on a fixed Riemann surface of given genus. This shows that for some function $f$ we have

$$\text{ind}(\mathcal{J}_\mathcal{L}) = f(\text{deg} \mathcal{L}, g).$$

The following computation is to determine the function $f$. Before we do so, let’s quickly prove Serre duality. Integrating by parts shows that

$$\langle \partial s, t \rangle = \langle s, \partial^\dagger t \rangle$$

where

$$\partial^\dagger : t \mapsto -g^{-1}h^{-1}\partial_z(ht).$$

Hence, $t \in H^1(X, \mathcal{L}) \iff t \in \ker \partial^\dagger \iff h^{-1}t \in \ker \partial$. Thus,

$$H^1(X, \mathcal{L}) = h^0(X, \mathcal{L}^{-1} \otimes \Lambda^{1,0}).$$

Consider the Riemann surface $X$ of the equation

$$f(z) = \sqrt{z(z-1)(z-2)\ldots(z-2g)}.$$

The cut locus for this function will be the intervals $[0,1], [2,3], \ldots, [2g, \infty]$. Glueing two copies of $\mathbb{C}$ together via these cuts and completing the surface will give a Riemann surface of topological genus $g$ that has a degree two cover of $\mathbb{P}^1$, denoted by the function $z$. It is easily seen that if a Riemann surface is a degree $d$ cover, then the field extension of meromorphic functions is at most $d$ (Showing that it is at least $d$ is hard, but follows quickly from Riemann-Roch). Thus, it is clear that

$$\mathbb{C}(X) = \mathbb{C}(z, f).$$

By rationalizing the denominator, we may write any element of $\mathbb{C}(X)$ as

$$\frac{a(z) + b(z)f(z)}{c(z)}$$

for some polynomials $a(z), b(z), c(z)$ with $c(z) \neq 0$. Thus, we define $\mathbb{C}(X)$ to be the field of fractions of $\mathbb{C}[z]$.
for relatively prime polynomials $a, b,$ and $c$.

We claim that the poles are the roots of $c$. Suppose the numerator vanished at a root of $c$. Let $z$ and $z'$ be the inverse images of a non-branched root in $\mathbb{P}^1$. If $a(z) + b(z)f(z) = a(z') + b(z')f(z') = 0$ then $b(z) = 0$, implying $a(z) = 0$ contradicting relative primality. If we are at a branched point, then $c(z)$ has a double root while $f(z)$ has a single root. Hence, $b(z) = 0$ as before, again implying that $a(z) = 0$, a contradiction.

Consider the space $H^0(d[0])$, which corresponds to elements of $\mathbb{C}(X)$ with a pole of order at worst $d$ at 0. Then, we may almost always assume that $c(z) = z^k$ where $k = \lfloor d/2 \rfloor$. The only case in which $k$ can be $\lceil d/2 \rceil$ is when $d$ is odd and $2g - 2 < d$, in which case we have a holomorphic section

$$\frac{f(z)}{z^k}$$

To ensure that there is no pole at $\infty$, we need

$$2k \geq \{2 \deg a, 2 \deg b + 2g + 1\}.$$ 

The number of such monomials $a$ and $b$ thus gives $h^0(d[0])$, except for when $d > 2g - 2$ is odd, when we must add 1. Thus,

$$h^0(d[0]) = \begin{cases} 2k + 1 - g + \delta_{d, odd} = d + 1 - g & \text{if } 2g - 2 < d \\ k + 1 & \text{if } 0 \leq d \leq 2g - 2 \\ 0 & \text{if } d < 0 \end{cases}.$$ 

We can explicitly construct a holomorphic section of $\Lambda^{0,1}$

$$\omega := \frac{z^{g-1}dz}{f}.$$ 

The divisor of $\omega$ is $(2g - 2)[0]$. Hence $h^0(d[0]^{-1} \otimes \Lambda^{1,0}) = h^0((2g - 2 - d)[0])$. Combining this with our above formula shows that on the hyperelliptic curve

$$h^0(d[0]) - h^0(d[0]^{-1} \otimes \Lambda^{1,0}) = d + 1 - g.$$ 

**Theorem 1.7.1** (Riemann-Roch). Let $X$ be a compact Riemann surface. Then

$$h^0(X, \mathcal{L}) - h^0(X, \mathcal{L}^{-1} \otimes \Lambda^{1,0}) = \deg \mathcal{L} + 1 - g.$$ 

7
Appendix: Classification of Smooth Complex Line Bundles

Recall that the degree of a holomorphic line bundle is defined to be

\[ \deg(L) = \int_X \partial \overline{\partial} \log(h) \]

for some metric \( h \) on \( L \). Assume \( \deg(L) = \deg(L') \). Choose a smooth section \( s \) of \( L \) that vanishes on a finite set of points \( S \). Then, by Stokes' theorem,

\[ \int_X \partial \overline{\partial} \log(h) = \int_X \partial \overline{\partial} \log(h) + d\partial \overline{\partial} \log(s) \approx \oint_{B_r(S)} \partial \overline{\partial} \log(hs) \]

\[ = \oint_{B_r(S)} \partial \overline{\partial} \log(hs) \]

\[ = \oint_{C_r(S)} \partial \overline{\partial} \log(hs) \]

\[ \approx \oint_{C_r(S)} \partial \overline{\partial} \log(s). \]

Taking the limit as \( r \to 0^+ \) shows that

\[ \deg(L) = \lim_{r \to 0^+} \oint_{C_r(S)} \partial \overline{\partial} \log s := \sum_p \deg_p(s). \]

Note that \( \deg_p(s) \) is an integer, as it is the difference of values of \( \log \) at the same point. Consider a section \( s \) of \( L' \otimes L^{-1} \), which has degree 0. Then, there is a set of paths that start at a point \( p \) where \( \deg_p(s) = 1 \) and end at a point \( q \) where \( \deg_q(s) = -1 \). We claim that it is possible to homotope the section \( s \) so that the zeroes at \( p \) and \( q \) collide and annihilate.

There is a map from the disc \( D^2 \to X \) which contains only the zeroes \( p \) and \( q \). Trivializing \( s \) and restricting to the boundary, we get a map \( S^1 \to \mathbb{C}^* \). Since the zeroes are opposite in sign, the integral around \( S^1 \) of \( \frac{d}{dz} \) is zero, implying \( s|_{S^1} \) is null-homotopic in \( \mathbb{C}^* \). We may then homotope \( s(D^2) \) to be in the interior of \( s(S^1) \), where it does not vanish. Iterating, we get a section of \( L' \otimes L^{-1} \) which vanishes nowhere, showing that \( L' \otimes L^{-1} \) is trivial and \( L = L' \) smoothly.

The converse is simple. If \( L = L' \) smoothly, there is a smooth nonvanishing function which takes sections of one to sections of the other. Thus, the degrees of sections will be equal.

The above argument holds for non-holomorphic smooth complex line bundles in the following sense: We may redefine the degree to be \( \sum_p \deg_p(s) \) for some section \( s \) which vanishes on a finite set of points, and the above argument shows that this is well-defined. The result still holds. Thus, we have the following theorem:

**Theorem 2.0.2.** All complex line bundles of equal degree are smoothly isomorphic.
In particular, consider $g$ and given a complex tangent vector $u \in TM \otimes \mathbb{C}$ we can project it to $u^{1,0}$ or $u^{0,1}$ via the operators $\frac{1+iJ}{2}$ or $\frac{1-iJ}{2}$. Let $d = dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}$. Then, define

$$\partial_J f = \left( \frac{1 + iJ}{2}d \right) f := \frac{df}{dz}dz_J.$$

Suppose we are given an almost complex structure $J$ on the disc in the torus which is possible to extend to the torus, and let $\partial_J(0,0) = \frac{d}{dx} - i\frac{d}{dy}$. Define $J_t(x,y) = J(tx,ty)$ and extend $J_t$ in a way such that $\partial_{J_t} \to \partial_{J_0}$ where $\partial_{J_0} = \frac{d}{dx} - i\frac{d}{dy}$. We know that $\partial_{J_0}$ has a right inverse on smooth functions given by

$$P : \{c_{mn}\} \to \{(m-in)^{-1}c_{mn} : mn \neq 0\} + c_{00}\overline{z}.$$

By Sobolev embedding, choose $k$ so that $W^k \subset C^1$, and consider the operator

$$(\partial_{J_0}, \partial(0,0)) : W^k \to W^{k-1} \times \mathbb{C}.$$

This is surjective because $\partial_{J_0}(cz) = 0$ while $\partial(cz)(0,0) = c$. Since surjectivity is an open condition on Banach spaces,

$$(\partial_{J_t}, \partial(0,0)) : W^k \to W^{k-1} \times \mathbb{C}$$

is surjective for some $t > 0$. There exists $f \in C^1$ so that $\partial_{J_t}f = 0$ and $\partial f(0,0) = 1$. Then, in a small enough neighborhood of the origin, $f$ is a $C^1$ diffeomorphism.

We claim $f$ is smooth. Note that

$$\partial_J(g \circ f) = \partial_J f.$$

In particular, consider $g = f^{-1}$. Then, we may conclude that $\partial(f^{-1}) \circ f = \partial_J f$ is smooth. Thus, $\partial(f^{-1})$ is $C^1$. As above, we may explicitly solve the $\partial$ equation, finding a function $h \in C^2$ such that $\partial h = \partial(f^{-1})$. Then, $\partial(h - f^{-1}) = 0$ and $h - f^{-1}$ is at least $C^1$. But then, by the classical theorem in complex analysis, $h - f^{-1}$ is smooth. Hence, $f^{-1}$ is $C^2$, and thus so is $f$. By induction, $f$ must be $C^\infty$.

Using this construction on neighborhoods of points on a surface $(M,J)$, we may find smooth charts $f_p$ around each point such that $\partial_J(f_p) = 0$. Then, $\partial(f_p \circ f_p^{-1}) = 0$, implying that $\{f_p\}$ is a smoothly compatible atlas of charts endowing $M$ with a complex structure, and this complex structure induces the given almost complex structure $J$.

**Theorem 3.0.3** (Newlander-Nirenberg). *A smooth almost complex structure on a compact smooth surface is induced by a smoothly compatible complex structure.*

3 Appendix: The Newlander-Nirenberg Theorem

We define the notion of an almost complex structure as a smooth automorphism $J$ of the tangent bundle such that $J^2 = -I$. In particular, $J$ is a tensor of type $(1,1)$. Given a complex structure, we may define $J(p)$ to be (the differential of) multiplication by $i$ in a chart centered at $p$. Because the transition functions on a complex manifold $M$ are holomorphic, their differential at a point is an element of $\mathbb{C}^*$. Hence, commutativity of complex multiplication implies that $J(p)$ is well-defined.

We note that $J(p)$ has eigenvalues $\pm i$. So, when we extend the action of $J$ to $TM \otimes \mathbb{C}$ linearly, we get a decomposition into $+i$ and $-i$ eigenspaces

$$TM \otimes \mathbb{C} = TM^{1,0} \oplus TM^{0,1}$$

and given a complex tangent vector $u \in TM \otimes \mathbb{C}$ we can project it to $u^{1,0}$ or $u^{0,1}$ via the operators $\frac{1-iJ}{2}$ or $\frac{1+iJ}{2}$. Let $d = dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}$. Then, define

$$\partial_J f = \left( \frac{1 + iJ}{2}d \right) f := \frac{df}{dz}dz_J.$$

We claim $f$ is smooth. Note that

$$\partial_J(g \circ f) = \partial_J f.$$
4 The Abel-Jacobi Theorem

We can easily prove now that there are $g$ holomorphic one-forms on any Riemann surface.

$$1 - h^0(\Lambda^{1,0}) = h^0(\mathbb{C}) - h^0(\Lambda^{1,0}) = \deg(\mathbb{C}) + 1 - g = 1 - g.$$  

Hence $h^0(\Lambda^{1,0}) = g$. The Abel-Jacobi map $A : X \to \mathbb{C}^g/\Lambda_2g$ is defined by

$$x \mapsto \left( \int_{z_0}^z \omega_1, \ldots, \int_{z_0}^z \omega_g \right).$$

This is well-defined only up to the period lattice

$$\Lambda_2g = \text{span}\{A(z_0) : \text{loops with basepoint } z_0\}.$$  

We claim that it is possible to normalize $\omega_i$ so that $\int_{C_i} \omega_j = \delta_{ij}$. If $\left( \int_{C_i} \omega_j \right)$ had zero determinant, then we could find $\omega$ so that $\int_{C_i} \omega = 0$ for all $i$. But then, we note that

$$0 < i \int_X \omega \wedge \overline{\omega} = i \int_{\partial X_{cut}} \omega \wedge \overline{\omega} = i \int_{\partial X_{cut}} f\overline{\omega} = i \sum_k \left( \int_{C_k} \omega \int_{D_k} \overline{\omega} - \int_{D_k} \omega \int_{C_k} \overline{\omega} \right) = 0$$

which is a contradiction. Hence, the normalization is possible. Let $\Omega_{jk} = \int_{D_j} \omega_k$. Writing $\omega = \sum c_i \omega_i$, the above calculation shows that

$$0 < i \sum_{jk} (c_k c_j \Omega_{k} - \overline{c_k} c_j \Omega_{k}) = i \sum_{jk} \overline{c_k} c_j (\Omega_{k} - \Omega_{k}).$$

As above,

$$0 = \int_X \omega_j \wedge \omega_k = \sum_i \left( \int_{C_i} \omega_j \int_{D_i} \omega_k - \int_{D_i} \omega_j \int_{C_i} \omega_k \right) = \Omega_{jk} - \Omega_{kj}.$$  

So $\Omega_{jk} = \Omega_{kj}$ and

$$0 < \sum_{jk} \overline{c_k} c_j \text{Im}(\Omega_{jk}).$$

Thus $\text{Im}(\Omega)$ is positive definite and $\Omega$ is symmetric. So

$$\mathbb{R} \text{-span} \left( \int_{C_i} \omega_j, \int_{D_i} \omega_j \right) = \mathbb{R} \text{-span}(I_g) \oplus \mathbb{R} \text{-span}(\Omega) = \mathbb{R}^g \oplus i\mathbb{R} \text{-span} \text{Im} \Omega = \mathbb{C}^g$$

because $\text{Im}(\Omega)$ is positive-definite, hence of nonzero determinant. Thus, $\Lambda_2g$ is full.

**Theorem 4.0.4 (Abel’s Theorem).** $D$ is the divisor of a meromorphic function on $X$ if and only if $A(D) \equiv 0 \pmod{\Lambda_2g}$ and $\deg(D) = 0$.

**Proof.** The Riemann-Roch theorem implies that $h^0(\Lambda^{1,0} \otimes [p] \otimes [q]) = g + 1$ while

$$h^0(\Lambda^{1,0} \otimes [p]) = h^0(\Lambda^{1,0} \otimes [q]) = g.$$  

Thus there is a unique form $\omega_{pq}$ that has poles at $p$ and $q$ with residues 1 and $-1$ respectively (the residues must sum to zero) such that $\int_{C_i} \omega_{pq} = 0$. The last condition is enforced by subtracting a suitable holomorphic $1$-form. Let $f$ be a meromorphic function. Then, $df/f$ is a meromorphic form with integer residues and can thus be written as

$$\frac{df}{f} = \sum_r \omega_{p_r,q_r} + \sum_i b_i \omega_i.$$  

The a priori complex numbers $b_i$ are in fact integers, as can be seen by integrating around $C_i$ (the left-hand side will be an integer, and the middle term will be 0). We note that

$$A(D) = \sum_r \left( \int_{p_r}^{q_r} \omega_1, \ldots, \int_{p_r}^{q_r} \omega_g \right)$$

$$= \sum_r \left( \int_{D_r} \omega_{p_r,q_r}, \ldots, \int_{D_r} \omega_{p_r,q_r} \right)$$

10
where the second equality follows from the condition \( \int_{C} \omega_{pq} = 0 \). Hence,

\[
A(D) = \left( \int_{D_1} \frac{df}{\omega_1}, \ldots, \int_{D_s} \frac{df}{\omega_s} \right) - \sum_{i} b_i (\Omega_{1i}, \ldots, \Omega_{gi}) \in \mathbb{Z}^g \oplus \mathbb{Z} \Omega = \Lambda_{2g}.
\]

Conversely, suppose \( A(D) \in \Lambda_{2g} \) and \( \deg(D) = 0 \). Then, construct a meromorphic form

\[
\omega = \sum_{r} \omega_{p_r q_r} + \sum_{i} b_i \omega_i
\]

such that the residues of \( \omega \) match \( D \) (possible since \( \deg(D) = 0 \)), and \( b_i \) integers such that

\[
A(D) + \sum_{i} b_i (\Omega_{1i}, \ldots, \Omega_{gi}) \in \mathbb{Z}^g.
\]

Then, we claim that

\[
f(z) = \exp \left( \int_{z_0}^{z} \omega \right)
\]

is a well-defined meromorphic function with the desired zeroes and poles. This is true because \( \int_{D_i} \omega = 0 \) and \( \int_{C} \omega \in \mathbb{Z}^g \).

Note that this shows the Abel map is injective on \( X \) when \( g \neq 0 \). If \( A(p) = A(q) \), there would be a meromorphic function with a pole at \( p \) and a zero at \( q \), which is impossible, because it would define an isomorphism to the Riemann sphere.

**Corollary 4.0.1.** The holomorphic line bundles of degree zero are classified by \( \mathbb{C}^g / \Lambda_{2g} \).

**Proof.** Every line bundle \( L \) has a meromorphic section \( s \) by the Riemann-Roch theorem (too see this, tensor with a point bundle of high degree). All other sections differ by a meromorphic function, so \( \text{div}(s) \) is well defined (mod \( \Lambda_{2g} \)). Conversely, if two bundles \( L \) and \( L' \) have sections \( s \) and \( s' \) such that \( \text{div}(s) \equiv \text{div}(s') \) (mod \( \Lambda_{2g} \)), then their ratio satisfies \( \text{div}(s/s') \equiv 0 \) (mod \( \Lambda_{2g} \)). Hence, there is a meromorphic function \( f \) such that \( \text{div}(s/s') = \text{div}(f) \). But then, \( \text{div}(s/s'f) = 0 \) so \( L \otimes (L')^{-1} \) has a nonvanishing holomorphic section, implying \( L = L' \).

Finally, we prove every element of \( \mathbb{C}^g / \Lambda_{2g} \) corresponds to some line bundle. Consider the complex manifold \( X^g \). Define

\[
\mu(p_1, \ldots, p_g) = \sum_{i=1}^{g} (p_i - p_0) \in \mathbb{C}^g / \Lambda_{2g}.
\]

We claim that \( \mu \) has nonzero Jacobian at some point in \( X^g \). In local coordinates \((z_1, \ldots, z_g)\), we have

\[
\frac{\partial \mu}{\partial z_i} = \frac{\partial}{\partial z_i} \left( \int_{p_0}^{z} \omega_1, \ldots, \int_{p_0}^{z} \omega_g \right) = \left( \frac{\omega_1}{dz_1}, \ldots, \frac{\omega_g}{dz_i} \right).
\]

So in local coordinates the Jacobian matrix is

\[
J(\mu) = \begin{pmatrix}
\omega_1/dz_1 & \cdots & \omega_g/dz_1 \\
\vdots & \ddots & \vdots \\
\omega_1/dz_g & \cdots & \omega_g/dz_g
\end{pmatrix}.
\]

Choose a point \( p_1 \) where \( \omega_1 \) is nonzero. Add some scalar times \( \omega_1 \) to each of \( \omega_2, \ldots, \omega_g \) so that they vanish at \( p_1 \). Now, find a point \( p_2 \) where \( \omega_2 \) doesn’t vanish, and add some scalar times \( \omega_2 \) to each of \( \omega_3, \ldots, \omega_g \) so that they vanish at \( p_2 \). Continue until we have a set of points \((p_1, \ldots, p_g)\). Then in some basis \( J(\mu)(p_1, \ldots, p_g) \) is upper triangular with nonzero diagonal. Hence \( \mu \) has full rank at \((p_1, \ldots, p_g)\).

The image of \( \mu \) therefore contains an open subset of \( \mathbb{C}^g / \Lambda_{2g} \). Since an open subset of any torus additively generates it, we conclude that the Abel-Jacobi map is surjective.

\[\square\]

In fact \( \mu \) itself surjects from general properties of maps of compact complex manifolds, but this a bit tough to prove, and is not necessary for our purposes.
5 The Uniformization Theorem

Theorem 5.0.5. A simply connected Riemann surface $X$ is either $\mathbb{P}^1$, $\mathbb{C}$, or $\mathbb{D}$.

Proof. If $X$ is compact, then $X$ is genus zero. We claim that all complex spheres are biholomorphic. By the Riemann-Roch theorem, $h^0([p]) = 2$ while $h^0(\mathbb{C}) = 1$. Thus, there is a meromorphic function on $X$ with a single pole at $p$. This is a biholomorphism $X \to \mathbb{P}^1$.

Now suppose $X$ is noncompact. We show $X$ is biholomorphic to an open subset of $\mathbb{C}$. Exhaust $X$ by simply connected compact subsets $X_n$ that have smooth boundary. Consider the compact surface $Y_n$ attained by glueing two copies of $X_n$ along the boundary. The almost complex structure on $X_n$ may be extended to $Y_n$ smoothly. Then, by Newlander-Nirenberg theorem, $Y_n$ has the structure of a complex sphere. Hence, there is a biholomorphism $Y_n \to \mathbb{P}^1$. Composing with the stereographic projection, we get a holomorphic map $f_n : X_n \to \mathbb{C}$. Normalizing $f_n$ such that $f_n(p) = 0$ and $f_n'(p) = 1$. By Riemann mapping theorem, we may assume that $f_n(X_n)$ is a disc of some radius.

We claim that $f_n$ contains a convergent subsequence to an injective holomorphic function. We find a subsequence of $\{f_n \circ f_m^{-1} : n \geq m\}$ which converges on the disc $f_m(X_m)$ by Koebe’s compactness theorem, proven below. Taking a diagonal sequence, we conclude $\{f_n\}$ has a convergent subsequence to an injective holomorphic function $f : X \to \mathbb{C}$. If $f$ is surjective, we’ve shown $X$ is biholomorphic to $\mathbb{C}$. Otherwise, $f(X)$ is an open (by open mapping theorem), simply connected, strict subset of $\mathbb{C}$. So, by the Riemann mapping theorem, it is biholomorphic to $\mathbb{D}$. So we’re done.

\[ \square \]

Lemma 5.0.1 (Koebe’s Compactness Theorem). Let $S$ denote the set of injective holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that $f(0) = 0$ and $f'(0) = 1$. Then, $S$ is sequentially compact on $\mathbb{D}$.

Proof. Fix a sequence $\{f_n\} \subset S$. Let

$$ R_n := \sup_{R > 0} \{\mathbb{D}(R) \subset f_n(\mathbb{D})\}. $$

If $R_n > 1$, then $R_n f_n^{-1}$ contradicts the Schwarz lemma. So $R_n \leq 1$. Choose a point $x_n \in \partial\mathbb{D}(R_n) - f_n(\mathbb{D})$, and define $g_n = f_n/x_n$. Note that $\mathbb{D} \subset g_n(\mathbb{D})$. Since $g_n(\mathbb{D})$ is simply connected and doesn’t contain 1, we can define $h_n = \sqrt{g_n - 1}$ holomorphically.

We claim that $h_n(\mathbb{D})$ doesn’t contain some disc $E$. To see why, note that $h_n(\mathbb{D}) \cap -h_n(\mathbb{D}) = \emptyset$. Indeed, if $h_n(z) = -h_n(z')$ then $g_n(z) = g_n(z')$ implying $z = z'$ by injectivity, but then $h_n(z) = h_n(z') = 0$ so $g_n(z) = 1$, a contradiction. So, we may apply some fixed inversion to get a sequence $i_n : \mathbb{D} \to \mathbb{D}$. Then, $i_n$ are uniformly bounded and therefore contain a convergent subsequence by Montel’s theorem. Thus $h_n$ and $g_n$ do too. Because $|x_n| \leq 1$, the sequence $f_n$ have a convergent subsequence to some function $f$ such that $f(0) = 0$ and $f'(0) = 1$.

Finally, we show that $f$ is injective. Note that for some $r < 1$ there is exists some $n$ so that

$$ \#\{f^{-1}(f(a))\} = \oint_{\mathbb{C}^*} \frac{f'(z) dz}{f(z) - f(a)} \approx \oint_{\mathbb{C}^*} \frac{f_n'(z) dz}{f_n(z) - f(a)} = 1 $$

by injectivity of $f_n$. By the integrality of the left-hand side $\#\{f^{-1}(f(a))\} = 1$, so $f$ is injective. \[ \square \]

Theorem 5.0.6 (Riemann Mapping Theorem). Every simply connected open domain $\Omega$ strictly contained in $\mathbb{C}$ is biholomorphic to $\mathbb{D}$.

Proof. Because $\Omega$ is strictly contained in $\mathbb{C}$ and simply connected, as in Koebe compactness, we may map $\Omega$ biholomorphically within the disc by taking a branch of the square root function, then inverting about a circle. So assume $\Omega \subset \mathbb{D}$. By applying a biholomorphism of $\mathbb{D}$ we may assume $0 \in \Omega$. We consider the family of all injective holomorphic functions $f : \Omega \to \mathbb{D}$ such that $f(0) = 0$. We define

$$ g(z) = \begin{cases} \frac{f(z)}{f'(0)} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}. $$

By maximum modulus principle, $g$ attains its maximum on the boundary of $\Omega$. Hence, $|f'(0)|$ is bounded above. Take a maximizing sequence $\{f_n\}$ for $|f'(0)|$. Because $\{f_n\}$ is uniformly bounded, Montel’s theorem
implies that it contains a convergent subsequence to a function \( f \). Note that \( f \) is injective by the second part of Koebe compactness, because \( f'_n(0) \) does not approach 0.

We claim that \( f \) surjects onto \( \mathbb{D} \). Suppose otherwise, i.e. \( p \notin f(\Omega) \). Consider the transformation

\[
T_p = \frac{z - p}{1 - \overline{p}z}
\]

which moves \( p \) to 0. Define

\[
g = T_{\sqrt{-p}} \circ \sqrt{\cdot} \circ T_p \circ f.
\]

Note that as before the square root can be defined because \( T_p(f(\Omega)) \) is simply connected and doesn’t contain 0. Note that

\[
s := T_p^{-1} \circ \sqrt{\cdot} \circ T_p^{-1},
\]

is not an automorphism of the disc, so by the Schwarz lemma, \( |s'(0)| < 1 \). By chain rule,

\[
f'(0) = (s \circ g)'(0) = s'(0)g'(0),
\]

contradicting maximality of \( |f'(0)| \). Hence \( f \) is surjective, as desired.

\[\square\]

**Lemma 5.0.2** (Montel’s Theorem, Weierstrass Theorem). A sequence of uniformly bounded holomorphic functions \( \Omega \to \mathbb{C} \) contains a convergent subsequence to a holomorphic function. Furthermore, the derivatives of this convergent subsequence converge to the derivative of their limit.

**Proof.** This is a standard fact in basic complex analysis which uses the Arzela-Ascoli theorem. \[\square\]
6 Maps of Hyperbolic Riemann Surfaces

We begin by proving the following result:

**Theorem 6.0.7.** Let $X$ and $Y$ be Riemann surfaces of genus 2 or more. There are finitely many nonconstant holomorphic maps from $X$ to $Y$.

**Proof.** Suppose $f_n$ is an infinite set of holomorphic maps from $X$ to $Y$. By Montel’s theorem, $f_n \to f$ uniformly with $f$ a holomorphic map from $X$ to $Y$. Cover $Y$ by small normal neighborhoods. For sufficiently large $n$, there is a unique time 1 geodesic $G(x,t)$ between $f(x)$ and $f_n(x)$ that lies completely in one neighborhood. Then, $s(x) = G'(x,0)$ defines a smooth section of $T_Y$.

Suppose that $s(p) = 0$. Then $f(p) = f_n(p)$. Consider a chart around $p$ in the disc gotten by the covering map $\mathbb{D} \to Y$. In this chart, the geodesics $G(x,t)$ are small circular segments. Define $H(x,t)$ as follows: Let $B_p$ be a bump function centered at $p$ inside this chart. Set

$$H(x,t) = B_p(f(x))(f(x)(1-t) + f_n(x)t) + (1 - B_p(f(x)))G(x,t).$$

Essentially $H(x,t)$ transitions from straight line segments near $p$ to geodesics outside of the support of $B_p$. Furthermore, $r(x) = H'(x,0)$ vanishes only where $s(x)$ vanishes. Note that in the chart around $p$,

$$r(x) = f_n(x) - f(x).$$

Hence, $r(x)$ has positive degree, as it is holomorphic in a neighborhood of its zeroes (note that $r(x)$ not actually holomorphic). But this is a contradiction, as $r(x)$ is a section of $f^*T_Y$, which has negative degree.

We now prove a strengthening of this result, originally due to Severi:

**Theorem 6.0.8.** Let $X$ be a Riemann surface of genus 2 or more. There are finitely many nonconstant holomorphic maps from $X$ to $Y_i$, where $Y_i$ is any Riemann surface of genus 2 or more.

**Proof.** By the Riemann-Hurwitz formula, $X$ may only map to Riemann surfaces of equal or smaller genus. Suppose infinitely many maps originate from $X$. Then for some fixed genus $h$, there are infinitely many maps from $X$ to Riemann surfaces of genus $h$. All such Riemann surfaces are smoothly equivalent.

7 Teichmüller Space

We know that any and all complex structures on a smooth surface $X$ arise from a smooth almost complex structure. Teichmüller space is defined as the space of complex structures up to biholomorphism homotopic to the identity map. If we further mod out by the mapping class group, we will get the moduli space $M_g$. By the Newlander-Nirenberg theorem, it is sufficient to study equivalence classes of smooth almost complex structures $[J]$.

Fix a complex structure $J_0$. Consider an different complex structure $J$, and expand it in the basis $\{\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\}$. In this basis $J$ will be of the form

$$J = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}^{-1} \begin{pmatrix} a & -b \\ \frac{1+\alpha^2}{b} & -a \end{pmatrix} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i(b + \frac{1+\alpha^2}{b}) & -2ai - b + \frac{1+\alpha^2}{b} \\ 2ai - b + \frac{1+\alpha^2}{b} & -i(b + \frac{1+\alpha^2}{b}) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \frac{\alpha^2 - 1}{\beta} & -\alpha \end{pmatrix}$$

for some $\alpha \in C^\infty(X, \mathbb{R})$ and $\beta \in C^\infty(X, \Lambda^{1,0} \otimes \Lambda_{0,1})$. By the Newlander-Nirenberg theorem, a coordinate $w$ for the structure $J$ exists, satisfying

$$\overline{\partial}Jw = \frac{1 + iJ}{2}dw = 0.$$

This differential equation may be expressed as

$$(1 + \alpha) \frac{dw}{dz} = i\beta \frac{dw}{dz}$$
and is called the Beltrami differential equation. A simple algebra check will show $|\beta| < 1 + \alpha$. Thus, there exists a $\mu \in C^\infty(X, \Lambda^{0,1} \otimes \Lambda_{1,0})$ with $||\mu||_\infty < 1$ such that $w$ satisfies

$$\frac{dw}{dz} = \mu \frac{dw}{dz}$$

written in the local coordinates of $J_0$. Conversely, given $||\mu||_\infty < 1$, we may choose an almost complex structure $J$ such that $\mu$ corresponds to it. So, we have converted our question about equivalence classes of smooth almost complex structures to analyzing the $L_\infty$ unit ball $B_1$ in the Beltrami differentials $C^\infty(X, \Lambda^{0,1} \otimes \Lambda_{1,0})$.

Before we proceed more formally, we sketch a way to describe the cotangent space of Teichmuller space at $[J]$. Consider a family of diffeomorphisms

$$\Phi_t(z) = z + t\xi + O(t^2)$$

with $\xi \in C^\infty(X, TX)$. Then, define the family $J(t) = d\Phi_t \circ J(0) \circ d\Phi_t^{-1}$ of equivalent complex structures. In Teichmuller space, $J'(0) = 0$, as the family $J(t)$ are all the same point. Computing, we get that

$$J(t) = \begin{pmatrix} * & 2i\partial \xi_{1,0}t \\ * & -i \end{pmatrix} + O(t^2).$$

Thus, the corresponding Beltrami equation is

$$\frac{dw}{dz} = \partial \xi_{1,0}t \frac{dw}{dz} + O(t^2).$$

Taking the derivative at $t = 0$, we see that $\mu \in T_J(B_1) \cong C^\infty(X, \Lambda^{1,0} \otimes \Lambda_{0,1})$ corresponds to the zero tangent vector at $[J]$ on Teichmuller space whenever

$$\mu = \partial \xi_{1,0}$$

for some $\xi_{1,0} \in C^\infty(X, \Lambda_{0,1})$. Hence, one would intuitively expect

$$T_{[J]} \text{Teich}(X) = \text{coker}(\partial|_{\Lambda_{1,0}}) = H^1(\Lambda_{1,0})^{\text{Serre Duality}} = (H^0(\Lambda_{1,0} \otimes \Lambda_{1,0}))^*. $$

This shows that the cotangent space of $\text{Teich}(X)$ at a point $[J]$ is the space of holomorphic quadratic differentials for $J$, which by Riemann-Roch has complex dimension $3g - 3$.

### 7.1 Quasiconformal Mappings

To define Teichmuller space formally, we use the Teichmuller metric. Suppose $J$ and $J'$ are smoothly compatible complex structures on $X$, with associated unit ball Beltrami differentials $\mu$ and $\mu'$. Let

$$d(J, J') = \log \inf_{f \in \text{Diff}_0(X, X)} \left\{ \max \left\{ \frac{|f_z| + |f_z^2|}{|f_z| - |f_z^2|} \right\} \right\}$$

where $\text{Diff}_0(X, X)$ are smooth diffeomorphisms homotopic to the identity map on $X$. This quantity exists, as we have already shown that the identity map from $X$ to itself satisfies $|f_z| \leq k|f_z|$ for $k < 1$. This inequality defines the notion of a quasiconformal map. Note that when $k = 0$, $f$ is a conformal, or holomorphic, map.

More fundamental than the quantity $k$ is

$$K := \frac{1 + k}{1 - k}$$

which has a natural geometric interpretation. A quasiconformal map will send small circles around $z$ to small ellipses around $f(z)$. The ratio of the major and minor axes of this ellipse is bounded by $K$. Furthermore, the factor $K(z)$ is multiplicative with composition of quasiconformal maps. In addition, $K = 1$ for conformal maps, showing that pre- and post-composition with conformal maps doesn’t change $K$. This motivates the
above definition $d(J, J')$, as it will now satisfy the triangle inequality, and will be invariant under pre- and post-composition with biholomorphisms. We would like to show that a quasiconformal map with minimal quasiconformal factor $d(J, J')$ is attained. Then, the distance between $J$ and $J'$ will equal zero if and only if they define conformally equivalent Riemann surfaces. Hence, the unit ball of Beltrami differentials will become Teichmüller space under metric identification via $d$.

**Theorem 7.1.1.** Let $f_n$ be a sequence of quasiconformal homeomorphisms between two compact Riemann surfaces $X$ and $Y$ with quasiconformal factors $K_n \to 1$. Then $f_n$ has a subsequence converging to a conformal map between $X$ and $Y$.

**Proof.** Note the similarity to Montel’s theorem. Lift the map to the universal covers of $X$ and $Y$ to get maps $f_n : \mathbb{D} \to \mathbb{D}$ and assume that $f(0)$ always lies in a single fundamental domain for $Y$. Then, we have a sequence of quasiconformal injective maps $f_n : \mathbb{D} \to \mathbb{D}$ satisfying certain periodicity conditions.

Suppose $f$ is $K$-quasiconformal. Then,

$$|f(b) - f(a)| \leq \int_a^b |f_z \, dz| + |f_{\bar{z}} \, d\bar{z}|$$

$$= \int_a^b (|f_z| + |f_{\bar{z}}|) \, ds$$

$$\leq \left( \int_a^b (|f_z|^2 - |f_{\bar{z}}|^2) \, ds \right)^{1/2} \leq \left( \int_a^b \left( \frac{|f_z|}{|f_{\bar{z}}|} - \frac{|f_{\bar{z}}|}{|f_z|} \right) \, ds \right)^{1/2} \cdot$$

The second inequality holds by Cauchy-Schwarz. Note that $|f_z|^2 - |f_{\bar{z}}|^2$ is the Jacobian. Choosing a length element $dt$ perpendicular to $ds$, and applying the quasiconformality condition, we get

$$|f(b) - f(a)| \leq \sqrt{c^{-1}K|b-a| \left( \int_a^b (|f_z|^2 - |f_{\bar{z}}|^2) \, ds \, dt \right)^{1/2} \leq \sqrt{\pi c^{-1}K|b-a|}$$

where $c$ is a constant such that the rectangle with corners $(a, b, a+c, b+c)$ lies within the unit disc. Thus, quasiconformal maps are locally Holder continuous.

Therefore, $f_n$ are equicontinuous and bounded. By the Arzela-Ascoli theorem, they admit a convergent subsequence to $f$. Since the inverse maps $f_n^{-1}$ are also quasiconformal with the same factor $K_n$ from the disc to itself, they too admit a convergent subsequence. It must be to $f^{-1}$ because $f_n$ converge to a non-boundary point at 0 and so by a continuity argument, $f_n^{-1}$ converge to $f^{-1}$. Furthermore, $f$ will be periodic in the same manner as all the $f_n$, so it will descend to a map from $X$ to $Y$.

Consider the case $K = 1$. Note that in $W^0(X)$ the $f_n$ converge to $f$ because the $f_n$ converge uniformly on compact subsets of $\mathbb{D}$ (in particular a fundamental domain for $X$). Hence, $\overline{D}f_n \to \overline{D}f \in W^{-1}(X)$ by continuity of the operator $\overline{D}$. But, we know that

$$||\overline{D}f_n||_{-1} \leq k_n ||\overline{D}f_n||_{-1} \leq k_n ||f_n||_0 \to 0$$

because $f_n$ is $K_n$ quasiconformal. Hence, $\overline{D}f = 0 \in W^{-1}(X)$. This implies that $f$ is actually holomorphic, as there is a pseudoinverse for $\overline{D}$. In particular, $f$ will lie in $W^k$ for all $k$, and so $f \in C^1$ and $\overline{D}f = 0$.

**7.2 Generalizing Quasiconformality**

Since holomorphic maps are smooth, we have proven that $d(J, J')$ is attained when $d(J, J') = 0$. It is important to note that we have not shown the infimum is always attained (i.e. in the cases $d(J, J') > 0$). The result is true, but only when we consider homeomorphisms in addition to diffeomorphisms. We need to modify our argument in this case: To define quasiconformality of homeomorphisms, we use the fact that all quadrilaterals are conformal to a rectangle with some fixed ratio of side lengths, then define the quasiconformal factor of a homeomorphism to be the supremum change in this quantity ranging over all quadrilaterals.
In accordance with our previous definition, Grotzsch’s theorem states that the minimal quasiconformal factor for a map of rectangles that fixes the corners is the ratio of the ratios of their sides. It is simple to prove with the notion of extremal length, which we made small use of in the above theorem.

**Lemma 7.2.1 (Grotzsch’s Theorem).** The minimal quasiconformal factor of a mapping of two rectangles is the ratio of the ratios of their sides.

**Proof.** Suppose the sides of the rectangles $R$ and $R'$ are $a, b,$ and $a', b'$. Consider all paths from the $a$ side to the opposite $a$ side and note that under the mapping, its length is at least $b'$. Using Cauchy-Schwarz as above

$$ab' \leq \int_0^a \int_0^b (|f_z| + |f_w|) \, ds \, dt \leq \sqrt{K}ab' \implies a/b : a'/b' \leq K.$$  

Furthermore, this value of $K$ is attained by the affine-linear map between the two rectangles, and this “distortion minimizing” map is unique. 

Teichmüller’s theorem is a natural generalization of Grotzsch’s theorem. There are holomorphic coordinates near almost all points on two Riemann surfaces such that the extremal quasiconformal mapping between them is affine linear. Such coordinates arise from the “measured foliations” of quadratic differentials—in these coordinates, the differentials are of the form $dz^2$. These special quadratic differentials are called the initial and terminal quadratic differentials of the extremal quasiconformal mapping.

The proof of Grotzsch’s theorem shows that there is no conformal equivalence between two rectangles whose sides have different ratios, but we still don’t know if every quadrilateral is conformally equivalent to one (yet). We say a quadrilateral is a Jordan curve with four marked points.

**Theorem 7.2.1.** Every quadrilateral is conformal to a unique complex rectangle.

**Proof.** Use the Riemann mapping theorem to map the interior of the quadrilateral to the upper half plane. The four corners will go to four points on the real line. Then, use the Schwarz-Christoffel mapping

$$z = f(\zeta) = \int_0^\zeta \prod_{i=1}^4 (w - w_i)^{-1/2} \, dw$$

which sends the boundary points $w_i$ to the corners of a polygon with angles $\pi/2$. To see why this works, note that in differential form,

$$dz = \left[ (w - w_1)(w - w_2)(w - w_3)(w - w_4) \right]^{-1/2} \, dw$$

and in particular,

$$\text{Arg}(dz) = -\frac{1}{2} \sum_i \text{Arg}(w - w_i) + \text{Arg}(dw).$$

Thus, going along the real line, $\text{Arg}(dz) = 0$ unless we pass by $w_i$ in which case the argument jumps by $-\frac{\pi}{2}$. The boundary of the half plane goes to the boundary of a polygon. Note that the region of analyticity includes all points on the real line except $w_i$, and one can show that the integral of $f'(w)/(f(w) - f(a))$ about semicircles centered at $w_i$ and $\infty$ go to zero for $a \in \mathbb{H}$. Thus

$$\# \{ f^{-1}(f(a)) \} = \int_{\mathbb{R}} \frac{f'(w) \, dw}{f(w) - f(a)} = \oint_{\partial P} \frac{dz}{z - f(a)} = \begin{cases} 1 & \text{if } f(a) \in P \\ 0 & \text{if } f(a) \notin P \end{cases}.$$  

So $f$ is a biholomorphism to $P$. 

Now, we are equipped to define quasiconformality of a homeomorphism. Let $m(Q)$ denote the module of a quadrilateral, that is to say, the unique value such that $\text{int}(Q)$ is conformal to a rectangle with side lengths $1, m(Q)$. Then, we define a $K$-quasiconformal map to be one such that

$$\frac{m(f(Q))}{m(Q)} \leq K$$

for all $Q$ whose closure lies in the domain. Reordering the vertices of $Q$ gives a quadrilateral $Q'$ such that $m(Q') = m(Q)^{-1}$. Hence $K \geq 1$. Furthermore, $f^{-1}$ is also $K$-q.c. and $K$ multiplies with composition.
**Definition 7.2.2.** Let $\Gamma$ be a family of rectifiable curves in $\Omega$. We say $\rho$ is an allowable function if $\rho$ is a nonnegative measurable function such that

$$A(\rho) := \int\int_{\Omega} \rho \, dx \, dy \in (0, \infty).$$

We define

$$L_\gamma(\rho) = \int_\gamma \rho \, ds$$

so long as this quantity is defined (otherwise we define it to be $\infty$). Let $L(\rho) = \inf_{\gamma \in \Gamma} L_\gamma(\rho)$. Then, we define the extremal length to be

$$\lambda(\Gamma) := \sup_{\rho} \frac{L(\rho)^2}{A(\rho)}.$$

**Exercise 7.2.1.** Compute $\lambda(\Gamma)$ for $\Gamma$ curves connecting opposite sides of a rectangle. Compute $\lambda(\Gamma)$ for $\Gamma$ the set of all curves connecting the inner and outer circle of an annulus. Hint: the extremal length is attained by $\rho = 1/r$.

**Theorem 7.2.3.** Let $\Omega \to \Omega'$ be $K$-q.c. and $\Gamma \to \Gamma'$. Then

$$K^{-1} \lambda(\Gamma) \leq \lambda(\Gamma') \leq K \lambda(\Gamma).$$

**Proof.** □

**Theorem 7.2.4.** A $K$-q.c. homeomorphism $f$ from the disc to itself satisfies

$$|f(b) - f(a)| \leq 16|b - a|^{1/K}$$

where $M$ depends only on $|f(0)|$ and $K$.

**Proof.** First apply a transformation which sets $f(0)$ to 0. □