Motivic Poincare series and knot homology

E. Gorsky

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Outline

Motivic Poincare series
  Poincare series and zeta functions
  Motivic Poincare series
  Irreducible case
  Properties
  General case: examples

Knot homology
  Heegard-Floer homology
  Conjectures
Poincare series and zeta functions

\( C = \bigcup_{i=1}^{r} C_i \) - plane curve singularity at the origin in \( \mathbb{C}^2 \).
\( \gamma_i : (\mathbb{C}, 0) \rightarrow (C_i, 0) \) - uniformisations of its components.

\[ v_i(f) = \text{Ord}_0(f(\gamma_i(t))) \]

One can define \( \mathbb{Z}^r \)-indexed filtration

\[ J_\nu = \{ f \in \mathcal{O} | v_i(f) \geq v_i \} \]

Consider the Laurent series

\[ L_C(t_1, \ldots, t_r) = \sum_{\nu} t_1^{\nu_1} \ldots t_r^{\nu_r} \cdot \dim J_\nu/J_{\nu+1}. \]
The Poincare series of the curve $C$ is defined by the formula

$$P_C(t_1, \ldots, t_r) = \frac{L_C(t_1, \ldots, t_r) \cdot \prod_{i=1}^{r} (t_i - 1)}{t_1 \cdot \ldots \cdot t_r - 1}$$

If $r = 1$, we get

$$P_C(t) = \sum_{v=0}^{\infty} t^v \cdot \dim J_v / J_{v+1}.$$

**Proposition** (Campillo, Delgado, Gusein-Zade)

$$P_C(t_1, \ldots, t_r) = \int_{\mathbb{P}^O} t_1^{v_1} \cdot \ldots \cdot t_r^{v_r} d\chi$$
$\Delta_C(t_1, \ldots, t_r)$ - multi-variable Alexander polynomial of the link of $C$

**Theorem (CDG)**

*If $r = 1$, then*

$$P_C(t)(1 - t) = \Delta_C(t),$$

*and if $r > 1$, then*

$$P_C(t_1, \ldots, t_r) = \Delta_C(t_1, \ldots, t_r).$$
Motivic Poincare series

In analogy to the construction of the motivic measure on the space of arcs, one can define a motivic measure on the ring $\mathcal{O}$ approximating it by jet spaces. Motivic Poincare series is the motivic integral

$$P_g^C(t_1, \ldots, t_r) = \int_{\mathcal{O}} t_1^{\nu_1} \cdot \ldots \cdot t_r^{\nu_r} d\mu$$
Let $q = \mathbb{L}^{-1}$ be a formal variable. Let $h(\nu) = \text{codim} J_\nu$, and

$$L_g(t_1, \ldots, t_r, q) = \sum_{\nu \in \mathbb{Z}^r} \frac{q^{h(\nu)} - q^{h(\nu+1)}}{1 - q} \cdot t_1^{\nu_1} \ldots t_r^{\nu_r}.$$ 

**Theorem (CDG)**

$$P_g^C(t_1, \ldots, t_r; q) = \frac{L_g^C(t_1, \ldots, t_r) \cdot \prod_{i=1}^r (t_i - 1)}{t_1 \cdot \ldots \cdot t_r - 1}$$

If $r = 1$, we have

$$P_g^C(t) = \sum_{\nu = 0}^{\infty} t^\nu \cdot \frac{q^{\text{codim} J_\nu} - q^{\text{codim} J_{\nu+1}}}{1 - q}.$$
Irreducible case

One can prove that

\[ P_C(t) = 1 + t^{\sigma_1} + t^{\sigma_2} + t^{\sigma_3} + \ldots, \]

where \( \{0, \sigma_1, \sigma_2, \ldots\} \) form the semigroup of \( C \). Then

\[ P^C_g(t; q) = 1 + qt^{\sigma_1} + q^2t^{\sigma_2} + q^3t^{\sigma_3} + \ldots. \]

Example. \( C = \{x^3 = y^5\} \).

\[ P_C(t) = 1 + t^3 + t^5 + t^6 + t^8 + t^9 + \ldots = \frac{(1 - t^{15})}{(1 - t^3)(1 - t^5)}, \]

Therefore

\[ P^C_g(t; q) = 1 + qt^3 + q^2t^5 + q^3t^6 + q^4t^8 + q^5t^9 + \ldots. \]
Properties

The reduced motivic Poincare series is the power series
\[ \overline{P}_g(t_1, \ldots, t_r) = (1 - qt_1) \cdot \ldots \cdot (1 - qt_r) \cdot P_g(t_1, \ldots, t_r). \]

Theorem (-)

1. **Polynomiality.** \( \overline{P}_g(t_1, \ldots, t_n; q) \) is a polynomial in \( t_1, \ldots, t_n \) and \( q \).

2. **Reduction to the Alexander polynomial.** If \( n = 1 \), then
\[ \overline{P}_g(t; q = 1) = \Delta(t), \]
where \( \Delta \) denote the Alexander polynomial of the link of the corresponding plane curve singularity. If \( n > 1 \), then
\[ \overline{P}_g(t_1, \ldots, t_n; q = 1) = \Delta(t_1, \ldots, t_n) \cdot \prod_{i=1}^{n} (1 - t_i). \]
3. **Forgetting components.** Let $C$ be a curve with $n$ components, and $C_1$ be an irreducible curve. Then

$$
\overline{P}_g^{C \cup C_1}(t_1, \ldots, t_n, t_{n+1} = 1) = (1 - q)\overline{P}_g^C(t_1, \ldots, t_n).
$$

If $C$ has only one component, then

$$
\overline{P}_g^C(t = 1) = 1.
$$

4. **Symmetry.** Let $\mu_\alpha$ be the Milnor number of $C_\alpha$, $(C_\alpha \circ C_\beta)$ is the intersection index of $C_\alpha \circ C_\beta$, $\mu(C)$ is the Milnor number of $C$. Let

$$
l_\alpha = \mu_\alpha + \sum_{\beta \neq \alpha} (C_\alpha \circ C_\beta), \quad \delta(C) = (\mu(C) + r - 1)/2.
$$

Then

$$
\overline{P}_g\left(\frac{1}{qt_1}, \ldots, \frac{1}{qt_r}\right) = q^{-\delta(C)} \prod_{\alpha} t_{\alpha}^{-l_\alpha} \cdot \overline{P}_g(t_1, \ldots, t_r).
$$
General case: algorithm

For a proper everywhere set $P$ we define $\tilde{H}_P$ - explicitly given polynomial divisible by $\prod_{i \in E(P)} (1 - u_i)$

Theorem

For a proper everywhere set $P$ define the numbers $d_P(n)$ by the equation

$$H_P(u) = \sum_n d_P(n)u^n d_P(n) = \prod_{i \in P} [(1 - qu_i)^{k_i-p_i} - 1(1 - u_i)^{p_i-1}] \tilde{H}_P(u_1, \ldots, u_s).$$

Then

$$\overline{P}_g(t_1, \ldots, t_r) = \sum_{P \in \mathcal{P}} (-1)^{|P|} q^{|P|} t_P \times \sum_n d_P(n)t^{Mn} q^{F(n)-\sum n_i}.$$
General case: examples

Consider the singularity $x^{k_0} - y^{k_0} = 0$.
For $0 < k < k_0$ let the numbers $c_k(n)$ be defined by the equation

$$A_k(u) = \sum_{n=0}^{\infty} u^n c_k(n) = (1 - uq)^{k_0-k-1}(1 - u)^{k-1},$$

and for $k = 0$ let the numbers $c_0(n)$ be defined by the equation

$$A_0(u) = \sum_{n=0}^{\infty} u^n c_0(n) = \frac{(1 - uq)^{k_0-1} - u(u - q)^{k_0-1}}{1 - u}.$$

$$P_g(t_1, \ldots, t_{k_0}) = \sum_{K \subset \neq K_0} (-1)^{|K|} q^{|K|} t_K \sum_{n=0}^{\infty} c_{|K|}(n)(t_1 \ldots t_{k_0})^n q^{\frac{n(n+1)}{2}}.$$
For example, if $k_0 = 2$,

$$A_1(u) = 1, \quad A_0(u) = \frac{1 - uq - u(u - q)}{1 - u} = 1 + u,$$

so

$$P_g(t_1, t_2) = 1 - qt_1 - qt_2 + qt_1 t_2.$$ 

If $k_0 = 3$,

$$A_1(u) = 1 - qu, \quad A_2(u) = 1 - u, \quad A_0(u) = 1 + (1 - 2q - q^2)u + u^2,$$

so

$$P_g(t_1, t_2, t_3) = 1 - q(t_1 + t_2 + t_3) + q^2(t_1 t_2 + t_1 t_3 + t_2 t_3) +$$

$$q(1 - 2q - q^2)t_1 t_2 t_3 + q^3 t_1 t_2 t_3(t_1 + t_2 + t_3) -$$

$$q^3 t_1 t_2 t_3(t_1 t_2 + t_1 t_3 + t_2 t_3) + q^3 t_1^2 t_2^2 t_3^2.$$
Heegard-Floer homology were introduced by P. Ozsvath and Z. Szabo. To each link \( L = \bigcup_{i=1}^{r} K_i \) they assign the collection of homology groups \( \hat{HFL}_d(L, h) \), where \( d \) is an integer and \( h \) belongs to some \( r \)-dimensional lattice. They give a ”categorification” of the Alexander polynomial of \( L \): if \( r = 1 \), then

\[
\sum_h \chi(\hat{HFL}_*(L, h)) t^h = \Delta^s(t),
\]

where \( \Delta^s(t) = t^{-\deg \Delta/2} \Delta(t) \) is the symmetrized Alexander polynomial of \( L \). If \( r > 1 \), then

\[
\sum_h \chi(\hat{HFL}_*(L, h)) t^h = \prod_{i=1}^{r} (t_i^{1/2} - t_i^{-1/2}) \cdot \Delta^s(t_1, \ldots, t_r).
\]
Theorem (Ozsvath, Szabo)
Let $g(K)$ be the genus of a knot $K$, i.e. the minimal genus of a Seifert surface for $K$. Then

$$g(K) = \max\{n | \dim \widehat{HFL}_*(K, n) \neq 0\}$$

Theorem (Ni)
A knot $K$ is fibered if and only if

$$\dim \widehat{HFL}_*(L, g(K)) = 1.$$
Consider the ring \( R = \mathbb{Z}[U_1, \ldots, U_r] \). For every \( r \)-component link \( L \) there exists a \( \mathbb{Z}[r] \)-filtered chain complex \( CFL^{-}(S^3, L) \) of \( R \)-modules, whose filtered homotopy type is an invariant of the link \( L \). The operators \( U_i \) lowers the homological grading by 2 and the filtration level by 1.

\[
\widehat{CFL}(S^3, L) = CFL^{-}(S^3, L)/(U_1 = \ldots = U_r = 0)
\]

\[
H^*(CFL^{-}(S^3, L)) = H^*(CFL^{-}(S^3)) = \mathbb{Z}[U]
\]

\[
H^*(CFL^{-}(S^3, L, k)/CFL^{-}(S^3, L, k - 1)) = HFL^{-}(S^3, L, k)
\]

\[
H^*(\widehat{CFL}(S^3, L, k)/\widehat{CFL}(S^3, L, k - 1)) = \widehat{HFL}(S^3, L, k)
\]
Let $K$ be the link of an irreducible curve singularity $C$. Consider the Poincare polynomial for the Heegard-Floer homologies:

$$HFL_C(t, u) = \sum u^d t^s \dim \widehat{HFL}_{d,s}(K).$$

It categorifies the Alexander polynomial in the sense that

$$HFL_C(t, -1) = t^{-\deg \Delta/2} \Delta_C(t).$$

**Theorem (-)**

Take $\overline{P}_g^C(t, q)$ and let us make a following change in it: $t^\alpha q^\beta$ is transformed to $t^\alpha u^{-2\beta}$, and $-t^\alpha q^\beta$ is transformed to $t^\alpha u^{1-2\beta}$. We get a polynomial $\tilde{\Delta}_g^C(t, u)$. Then

$$\tilde{\Delta}_g^C(t^{-1}, u) = t^{-\deg \Delta/2} HFL_C(t, u). \tag{1}$$
Key lemma

Suppose that a cochain complex $C$ has a filtration $C_k$, $k \geq 0$ and an injective operator $U$ of homological degree 2 acting on it such that
1) $U(C_k) \subset C_{k+1}$ and $U^{-1}(C_k) \subset C_{k-1}$
2) $H^*(C_k / U(C_k))$ has rank 1 for all $k$,
Then for all $k$ the rank of $H^*(C_k / C_{k+1})$ is at most 1. Let
$\{0, \sigma_1, \sigma_2, \ldots\}$ is the set of $k$ such that this rank is 1. Then
3) $H^*(C_{\sigma_k} / C_{\sigma_k+1})$ belongs to degree $2k$. 
Key lemma cont’d

Let

\[ Q(t, q) = \sum_{k=0}^{\infty} q^k t^{\sigma_k}, \quad \overline{Q}(t, q) = Q(t, q)(1 - qt). \]

Let us make a following change in \( \overline{Q} \): \( t^\alpha q^\beta \) is transformed to \( t^\alpha u^{2\beta} \), and \(-t^\alpha q^\beta\) is transformed to \( t^\alpha u^{2\beta-1} \).

4) The result is equal to

\[ \sum_{k,n} t^k u^n \dim H^n(C_k/(C_{k+1} + UC_{k-1})). \]

The last result can be reformulated as follows. Consider the complex \( \hat{C}_k = C_k/UC_{k-1} \), then the last homology are the homology of the quotient \( \hat{C}_k/\hat{C}_{k-1} \). The multiplication by \( 1 - qt \) corresponds to the exact sequence

\[ 0 \to C_{k-1}/C_k \overset{u}{\to} C_k/C_{k+1} \to \hat{C}_k/\hat{C}_{k+1} \to 0. \]
Conjectures

N. Dunfield, S. Gukov and J. Rasmussen conjectured that all knot homology theories (Khovanov, Heegard-Floer, Khovanov-Rozansky) are parts, or specializations of a unified picture. They conjectured the existence of a triply-graded knot homology theory $\mathcal{H}_{i,j,k}(K)$ with the following properties:

- **Euler characteristic.** Consider the Poincare polynomial

  $$\mathcal{P}(K)(a, q, t) = \sum a^i q^j t^k \dim \mathcal{H}_{i,j,k}.$$ 

  Its value at $t = -1$ equals to the value of the reduced HOMFLY polynomial of the knot $K$:

  $$\mathcal{P}(K)(a, q, -1) = \mathcal{P}(K)(a, q).$$
Differentials. There exist a set of anti-commuting differentials $d_j$ for $j \in \mathbb{Z}$ acting in $\mathcal{H}_*(K)$. For $N > 0$, $d_N$ has triple degree $(-2, 2N, -1)$, $d_0$ has degree $(-2, 0, -3)$ and for $N < 0$ $d_N$ has degree $(-2, 2N, -1 + 2N)$.

Symmetry. There exists a natural involution $\phi$ such that

$$\phi d_N = d_{-N} \phi$$

for all $N \in \mathbb{Z}$.

Let

$$\mathcal{H}^N_{p,k}(K) = \bigoplus_{iN+j=p} \mathcal{H}_{i,j,k}(K).$$

Conjecture. There exists a homology theory with above properties such that for all $N > 1$ the homology of $(\mathcal{H}^N_*(K), d_N)$ is isomorphic to the $sl(N)$ Khovanov-Rozansky homology. For $N = 0$, $(\mathcal{H}^0_*(K), d_0)$ is isomorphic to the Heegard-Floer knot homology. The homology of $d_1$ are one-dimensional.
Consider "stable limit" of torus knots $T_{n,m}$ at $m \to \infty$.

$$P_s(T_n) = \lim_{m \to \infty} P_s(T_{n,m}) = \prod_{k=1}^{n-1} \frac{(1 - a^2 q^{2k})}{(1 - q^{2k+2})}. $$

**Conjecture** The limit homology $\mathcal{H}(T_n) = \lim_{m \to \infty} \mathcal{H}(T_{n,m})$ is a free polynomial algebra with $n - 1$ even generators with gradings $(0, 2k + 2, 2k)$ and $n - 1$ odd generators with gradings $(2, 2k, 2k + 1)$, therefore

$$P_s(T_n) = \prod_{k=1}^{n-1} \frac{(1 + a^2 q^{2k} t^{2k+1})}{(1 - q^{2k+2} t^{2k})}. $$
We denote the odd generators by $\xi_1, \ldots, \xi_{n-1}$, and even generators by $e_1, \ldots, e_{n-1}$.

The differentials send $\xi_k$ to some polynomials in $e_m$, and they are extended to the whole algebra by the Leibnitz rule. Taking into account the gradings, one can uniquely guess the equations

$$d_{-n}(\xi_k) = \delta_{k,n}, \quad d_0(\xi_k) = e_{k-1}, \quad d_1(\xi_k) = e_k.$$

The construction of the higher differentials is less restricted by the grading, however for small degrees one has no choice but to define

$$d_2(\xi_2) = e_1^2, \quad d_2(\xi_3) = e_1 e_2, \quad d_3(\xi_3) = e_1^3.$$
**Example.** Knot $T_{3,4}$ (link of $E_6$):

Vertical lines correspond to the differential $d_0$, its homology has dimension 5, as expected for Heegaard-Floer homology.