Cherednik algebras, Hilbert schemes and knot invariants

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Outline

I will present two different approaches to refined invariants of torus knots. They can be described:

I: As characters of certain representations of rational Cherednik algebras.

II: As matrix elements of certain operators in the polynomial representation of double affine Hecke algebras.
[I. Cherednik; E.G.-A.Negut]

The approaches are conjecturally equivalent. Their equivalence can be proved in some special cases and gives rise to interesting combinatorial identities that can be verified on a computer.
These two approaches match two approaches in mathematical physics:

I is expected to match the computations of the refined BPS invariants
[N. Dunfield-S. Gukov-J.Rasmussen; S.Gukov-M.Stosic]

II matches the refined Chern-Simons invariants
[M. Aganagic-S. Shakirov; A. Morozov et. al.]

Some connection to (colored) triply graded Khovanov-Rozansky homology is also expected.
(3,4) torus knot
Unrefined invariants

Unknot

Let us describe the unrefined invariants first. Given two coprime integers \((m, n)\) and a Young diagram \(\lambda\), one can construct \(\lambda\)-colored HOMFLY invariant \(P^\lambda_{m, n}(a, q)\) of the \((m, n)\) torus knot. At \(a = q^N\) it specializes to the corresponding \(sl(N)\) quantum invariant.

If \((m, n) = (1, 1)\), the knot is trivial and

\[
P^\lambda_{1, 1}(q^N, q) = P^\lambda(q^N, q) = s_\lambda(1, q, \ldots, q^{N-1}),
\]

where \(s_\lambda\) is a Schur polynomial. This determines the rational function \(P^\lambda(a, q)\) uniquely.
Unrefined invariants
Rosso-Jones formula

Theorem (M. Rosso-V. Jones)
One has

\[ P_{\lambda}^{\mu}(a, q) = \sum_{\mu} q^{-\frac{m}{n} \kappa(\mu)} c_{\lambda, n}^{\mu} P_{\mu}(a, q), \]

where \( \kappa(\mu) = \sum_{(i, j) \in \mu} (i - j) \) is the content of \( \mu \) and the coefficients \( c_{\lambda, n}^{\mu} \) are defined by the equation

\[ s_{\lambda}(x_1^n, x_2^n, \ldots) = \sum_{\mu} c_{\lambda, n}^{\mu} s_{\mu}. \]
Rational Cherednik algebras

Definition

Let $\mathfrak{h}_n$ denote the Cartan subalgebra for $\mathfrak{sl}(n)$. Rational Cherednik algebra $H_c$ of type $\mathfrak{sl}(n)$ is generated by $\mathfrak{h}$, $\mathfrak{h}^*$ and the group algebra $\mathbb{C}[S_n]$ modulo following relations;

$$[x, x'] = 0, \quad [y, y'] = 0,$$

$$[x, y] = (x, y) - c \sum_{s \in S} (\alpha_s^*, x)(\alpha_s, y)s,$$

where $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$, $S$ is the set of all reflections in $S_n$ and $\alpha_s$ is the equation of the fixed hyperplane for a reflection $s$. 
Given a representation $\pi_\lambda$ of $S_n$, one can define the action of $H_c$ on

$$M_c(\lambda) := \pi_\lambda \otimes \mathbb{C}[\mathfrak{h}].$$

Irreducible representations $L_c(\lambda)$ are simple quotients of $M_c(\lambda)$.

All representations of $H_c$ are naturally graded.

**Theorem (Y. Berest-P. Etingof-V. Ginzburg)**

The representation $L_c(\lambda)$ is finite-dimensional iff $\lambda = (n)$ and $c = m/n, \text{GCD}(m, n) = 1$. 
Rational Cherednik algebras

Characters

Let $m$ and $n$ be coprime integers, $\lambda$ be a Young diagram with $d$ boxes. Define a space

$$\mathcal{H}_{m,n}^\lambda = \text{Hom}_{S_{nd}}(\Lambda \cdot \mathfrak{h}, L_{\frac{m}{n}}(n \lambda)).$$

It has $q$-grading induced from the grading on $L_{\frac{m}{n}}(n \lambda)$ and $a$-grading induced from the exterior degree on $\Lambda \cdot \mathfrak{h}$.

**Theorem**

a) [E.G.-A. Oblomkov, J. Rasmussen-V.Shende] The bigraded character of $\mathcal{H}_{m,n}^{(1)}$ equals to $P_{m,n}^{(1)}$.

b) [P. Etingof-E.G.-I. Losev] The bigraded character of $\mathcal{H}_{m,n}^\lambda$ equals to $P_{m,n}^\lambda$. 
Rational Cherednik algebras

Filtrations

Following the ideas of I. Gordon, in [GORS] we introduced a filtration on $L_{m/n}(n)$ with many useful properties. The character of the associated graded space to $\mathcal{H}_{m/n}$ is a 3-variable deformation of the HOMFLY polynomial. A filtration for general $\lambda$ has yet to be constructed.

In this example we see a filtration on a 16-dimensional representation $L_{4/3}(3)$. The corresponding space $\mathcal{H}_{3,4}(3)$ has dimension 11.
The double affine Hecke algebra (DAHA) was introduced by I. Cherednik and played an important role in his proof of Macdonald’s conjectures. We will use DAHA $A_N$ of type $GL_N$. Its main features are:

- It contains two commutative subalgebras $\mathbb{C}[X_1, \ldots, X_N]$ and $\mathbb{C}[Y_1, \ldots, Y_N]$.
- It enjoys the action of $SL(2, \mathbb{Z})$ by algebra automorphisms.
- It has a polynomial representation $V_N = \mathbb{C}[X_1, \ldots, X_N]$: $X_i$ act by multiplication operators, and $Y_i$ act by certain difference operators. In particular, $Y_1 + \ldots + Y_N$ is the Macdonald’s operator $D$. 
Following the physical constructions of M. Aganagic and S. Shakirov, I. Cherednik defined a remarkable family of polynomials by the following procedure:

- Let $M_\lambda$ be a Macdonald polynomial in $X_i$, considered as an element in $A_N$.
- Pick $K_{m,n} \in SL(2, \mathbb{Z})$ such that $K_{m,n}(1,0) = (m,n)$.
- Define $W_{m,n}^\lambda := K_{m,n}(M_\lambda) \in A_N$.
- Evaluate $W_{m,n}^\lambda$ in the polynomial representation:

$$P_{m,n}^{\lambda,N}(q,t) = \varepsilon_N(W_{m,n}^\lambda(1)),$$

where $\varepsilon_N(f) = f(1,\ldots,q^{N-1})$. 


Theorem (E.G.-A.Negut)

a) There is a polynomial $P_{m,n}^\lambda(a, q, t)$ such that

$$P_{m,n}^{\lambda,N}(q, t) = P_{m,n}^\lambda(q^N, q, t).$$

b) $P_{n,m}^{(1)}(a, q, t) = \frac{1}{(q-1)(t-1)} \times$

$$\times \sum_{\lambda=\square_1 + \ldots + \square_n}^{\text{SYT}} \prod_{i=1}^{n} \chi_i^{S_{m/n}(i)} \cdot \frac{a-\chi_i}{1-\chi_i} \prod_{1 \leq i < j \leq n} \omega \left( \frac{\chi_i}{\chi_j} \right)$$

where the sum is over all standard Young tableaux of size $n$, $\omega(x) = \frac{(x-1)(x-qt)}{(x-q)(x-t)}$, the constant $\chi_i$ denotes the $q, t$-content of the box $\square_i$ and: $S_{m/n}(i) = \left[ \frac{im}{n} \right] - \left[ \frac{(i-1)m}{n} \right]$. 
O. Schiffmann and E. Vasserot proved that there is a well-defined $N \to \infty$ limit $\mathbf{SH}$ of the symmetrized algebras $\mathcal{A}_N$, which may be called "spherical DAHA for $GL_\infty". The Cherednik's operators and the polynomial representation can be appropriately renormalized and considered in this limit.

The algebra $\mathbf{SH}$ is related to other interesting algebras: elliptic Hall algebra, shuffle algebra of Feigin-Odesskii.

Schiffmann-Vasserot and Feigin-Tsymbalyuk constructed the action of $\mathbf{SH}$ on the equivariant K-theory of the Hilbert schemes of points on $\mathbb{C}^2$. A. Negut realized the operators $W_{\lambda m, n}^\lambda$ as certain geometric correspondences.

The formula for $P_{n,m}^{(1)}(a, q, t)$ comes from the equivariant localization of his construction.
Up to some regrading, we have the following expression for $P_{3,4}^{(1)}$ as sum over 4 standard tableaux of size 3:

$$
\frac{t^6}{(t - q)(t^2 - q)}(1 + \frac{a}{t})(1 + \frac{a}{t^2}) - \frac{q^2 t^2}{(q - t)(q^2 - t)}(1 + \frac{a}{t})(1 + \frac{a}{q})
$$

$$
- \frac{q^2 t^2}{(q - t)(q - t^2)}(1 + \frac{a}{t})(1 + \frac{a}{q}) + \frac{q^6}{(q - t)(q^2 - t)}(1 + \frac{a}{q})(1 + \frac{a}{q^2})
$$

$$
= q^3 + qt + q^2 t + qt^2 + t^3 + a (q + q^2 + t + qt + t^2) + a^2.
$$

The right hand side has 11 terms and matches the filtered character of $H_{3,4}^{(1)}$. 
The relation between two approaches is expected to be of geometric nature: both are related to certain sheaves on the punctual Hilbert scheme of points on $\mathbb{C}^2$.

- A. Negut’s geometric construction realizes the operator $W_{n,m}^{(1)}$ as a certain geometric correspondence on $Hilb^i \mathbb{C}^2 \times Hilb^{n+i} \mathbb{C}^2$. In particular, $W_{n,m}^{(1)}(1)$ is a vector in $K(Hilb^n \mathbb{C}^2)$ realized by a certain explicit sheaf $\mathcal{F}_{m/n}$. The refined knot invariant can be computed as

$$\mathcal{P}(a, q, t) = \int_{Hilb^n \mathbb{C}^2} \mathcal{F}_{m/n} \otimes \bigoplus_{i=0}^{n} a^i \wedge^i T$$

where $T$ is the tautological rank $n$ bundle on $Hilb^n \mathbb{C}^2$.

- Results of I. Gordon and T. Stafford imply that to any (suitably filtered) representation of a rational Cherednik algebra one can associate a sheaf on $Hilb^n \mathbb{C}^2$. 


Connection
Example: $m = n + 1$

The case $m = n + 1$ was studied in details.

- The sheaf $\mathcal{F}_{(n+1)/n}$ coincides with the restriction of the line bundle $\Lambda^n T$ to the punctual Hilbert scheme.

- It was studied by M. Haiman in connection with his work on diagonal harmonics. In particular, $P_{n,n+1}^{(1)}(a = 0, q, t)$ coincides with the $q, t$-Catalan numbers of Garsia and Haiman.

- On the other hand, I. Gordon constructed a filtration on $L_{(n+1)/n}$ and showed that it is related to $\mathcal{F}_{(n+1)/n}$ under the Gordon-Stafford correspondence.
Thank you.