Finding real roots of Eisenstein series of weight one by Painlevé VI equations

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Abstract

For any $(r,s) \in \mathbb{C}^2 \backslash \frac{1}{2}\mathbb{Z}^2$, we define $p_{r,s}(\tau) \in \mathbb{C}$ by

$$\wp(p_{r,s}(\tau)|\tau) = \wp(r+s\tau|\tau) + \frac{\wp'(r+s\tau|\tau)}{2(\zeta(r+s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau))}, \quad \tau \in \mathbb{H}$$

The Hitchin theorem says that $p_{r,s}(\tau)$ is a solution of a Painlevé VI equation. Remarkably, the denominator $\zeta(r+s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau)$ is the Eisenstein series of weight one if (r,s) is an N-torsion point.

Set $Z_{r,s}(\tau) = \zeta(r+s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau)$. Then $Z_{r,s}(\tau)$ has another link with geometric and PDE aspects. Any zero τ of $Z_{r,s}(\tau)$ for some real pair $(r,s) \in \mathbb{R}^2 \backslash \frac{1}{2}\mathbb{Z}^2$, the PDE (mean field equation)

$$\Delta u + e^u = 8\pi\delta(0) \text{ in } E_\tau,$$

has a solution, where E_τ is the torus $\mathbb{C}/\wedge_\tau$, $\wedge_\tau$ is the lattice generated by 1 and τ, and $\delta(0)$ is the Dirac measure at lattice points. In this talk, I will give a survey about the story about the connection among such subjects: mean field equations, Painlevé VI, and modular forms. The center of these connections are Green functions and Lame equations.