Finding real roots of Eisenstein series of weight one by Painlevé VI equations

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Abstract

For any \((r, s) \in \mathbb{C}^2 \setminus \frac{1}{2} \mathbb{Z}^2\), we define \(p_{r,s}(\tau) \in \mathbb{C}\) by

\[
\wp(p_{r,s}(\tau)) = \wp(r + s\tau|\tau) + \frac{\wp'(r + s\tau|\tau)}{2(\zeta(r + s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau))}, \quad \tau \in \mathbb{H}
\]

The Hitchin theorem says that \(p_{r,s}(\tau)\) is a solution of a Painlevé VI equation. Remarkably, the denominator \(\zeta(r + s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau)\) is the Eisenstein series of weight one if \((r, s)\) is an \(N\)-torsion point.

Set \(Z_{r,s}(\tau) = \zeta(r + s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau)\). Then \(Z_{r,s}(\tau)\) has another link with geometric and PDE aspects. Any zero \(\tau\) of \(Z_{r,s}(\tau)\) for some real pair \((r, s) \in \mathbb{R}^2 \setminus \frac{1}{2} \mathbb{Z}^2\), the PDE (mean field equation)

\[
\Delta u + e^u = 8\pi \delta(0) \text{ in } E_\tau,
\]

has a solution, where \(E_\tau\) is the torus \(\mathbb{C}/\Lambda_\tau\), \(\Lambda_\tau\) is the lattice generated by 1 and \(\tau\), and \(\delta(0)\) is the Dirac measure at lattice points. In this talk, I will give a survey about the story about the connection among such subjects: mean field equations, Painlevé VI, and modular forms. The center of these connections are Green functions and Lame equations.