A quantitative description of Hawking radiation.

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Quantum field theory

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- Quantum fields are wave function functionals: $\mathbb{E} : \psi \mapsto \mathbb{E}(\psi)$. 

If the particle dynamics is given by a propagator $U(t,0)$, i.e. $\psi_t = U(t,0)\psi_0$ then the state dynamics must satisfy:

$$\mathbb{E}_t(\psi_t) = \mathbb{E}_0(\psi_0) \Leftrightarrow \mathbb{E}_t(U(0,t)\psi_t) = \mathbb{E}_0(\psi_0) \Leftrightarrow \mathbb{E}_t(\psi_t) = \mathbb{E}_0(U(0,t)\psi_t).$$

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This reduces the analysis of quantum fields to (a) a PDE problem and (b) a (possibly difficult) computation.
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g = \frac{\Delta_r}{r^2} dt^2 - \frac{r^2}{\Delta_r} dr^2 - r^2 d\sigma_{S^2}(\omega)
\]

\[
\Delta_r = r^2 \left( 1 - \frac{\Lambda r^2}{3} \right) - 2M_0 r, \quad \Lambda, M > 0
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▶ This metric can be extended beyond the horizons $r = r_+$ and $r = r_-$.

▶ The surface gravities of the black hole and cosmological horizons are characteristic parameters given by:

$$\kappa_{\pm} = \frac{|\Delta_r'(r_{\pm})|}{2r_{\pm}^2} .$$
Collapsing star in SdS

We set another system of coordinates $S_\ast$ by $(t, x, \omega)$ with

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Radial geodesics propagate along $t \pm x = \text{cte}$ and $r_+, r_-$ get send to $+\infty$ and $-\infty$, respectively.
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- We want to study quantum fields in this space. We need an evolution equation for particles.
The evolution equation

- We consider spin-0 particles with mass $m$ in the Schwarzschild–de Sitter spacetime. The equation is given by

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We put reflecting boundary conditions on the collapsing star. We study the backward propagation starting at time $T \to +\infty$:

$$\begin{cases} 
(\Box_g + m^2)u = 0 \\
|u|_B = 0 \\
(u, \partial_t u)(T) = (u_0, u_1).
\end{cases}$$

This is the mathematical basis for Hawking radiation.
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- We will focus only on (a) in this talk.
Asymptotic of scalar fields

**Theorem [D '17]**

Consider $u_0, u_1$ smooth with compact support, and $u$ solution of

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There exist scattering fields (see later) $u_-, u_+$ smooth and exponentially decaying; and $c_0 > 0$ such that for $t$ near 0,

\[
u(0,x,\omega) = r_- r u_-(\frac{1}{\kappa} - \ln(x e^{-\kappa - T})), \omega) + u_+(T-x,\omega) + O(\mathcal{H}^1/2(e^{-c_0 T}).
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($\kappa$ is the surface gravity of the black-hole.)
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\[
u(0, x, \omega) = \frac{r_-}{r} u_-(\frac{1}{\kappa_-} \ln \left( \frac{x}{e^{-\kappa_- T}} \right), \omega) + u_+(T - x, \omega) + O_{\mathcal{H}^{1/2}}(e^{-c_0 T}).
\]

($\kappa_-\) is the surface gravity of the black-hole.)
\[ u_0, u_1 \]

\[ t = T \]

\[ \frac{r_-}{r} u_- \left( \frac{1}{\kappa_-} \ln \left( \frac{x}{e^{-\kappa_- T}} \right) \right) \]

\[ u_+ (T - x) \]
Comments

- The black hole temperature \( \kappa_-/(2\pi) \) emerges.
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Thus the result gives exponential convergence to equilibrium. The rate $c_0$ can be computed explicitly: it depends only on $\kappa_-$, $\kappa_+$ and the first resonance of the K–G equation on the black-hole background.
The Hawking effect

Let $\mathcal{F}^{\mathcal{H},\beta}$ the Bose–Einstein state at temperature $1/\beta$ with respect to a Hamiltonian $\mathcal{H}$.

Interpretation: at time 0, the quantum state is that of a Bose–Einstein gas with cosmological background temperature $\kappa + \frac{2\pi}{\beta}$.

As time goes, this state splits to two Bose–Einstein states with respect to the asymptotic Hamiltonians $\mathcal{H}_x$.

The first one sees no change in temperature while the second one acquires the black-hole temperature $\kappa - \frac{2\pi}{\beta}$.
The Hawking effect

- Let $|\mathbb{E}^{\mathbb{H},\beta}\rangle$ the Bose–Einstein state at temperature $1/\beta$ with respect to a Hamiltonian $\mathbb{H}$.

- Let $\mathbb{H}_0$ be the black-hole Klein–Gordon Hamiltonian in $S_*$: the K–G equation takes the form $(\partial_t^2 - \mathbb{H}_0)u = 0$. 

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- Let $E_{\mathcal{H}, \beta}$ the Bose–Einstein state at temperature $1/\beta$ with respect to a Hamiltonian $\mathcal{H}$.
- Let $\mathcal{H}_0$ be the black-hole Klein–Gordon Hamiltonian in $S_*$: the K–G equation takes the form $(\partial_t^2 - \mathcal{H}_0)u = 0$.
- Thanks to the theorem:

$$E_{\mathcal{H}_0, 2\pi/\kappa^+}(U(0, T)(u_0, u_1)) = E_{D_x^2, 2\pi/\kappa^+}(u_+, D_x u_+).E_{D_x^2, 2\pi/\kappa^-}(u_-, D_x u_-).\left(1 + O(e^{-c_0 T})\right).$$
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Previous related results

- Bachelot late ’90s, Melnyk early ’00s – emission of bosons and of fermions by Schwarzschild black holes and Schwarzschild–de Sitter black holes.
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- This work provides the first rates of convergence. The previous proofs were not fully constructive.
- We take full advantage of recent decay results for waves in black hole spacetimes. For the dS black-holes, see Bachelot–Motet-Bachelot ’93, Sa-Barreto–Zworski ’97 (resonances), Bony–Häfner ’07 (exponential decay), Dafermos–Rodnianski ’07 (polynomial decay), Melrose–Sa-Barreto–Vasy ’08, Vasy ’13 (geometric methods), Dyatlov ’11 –’ 12 (rotating black holes), Hintz–Vasy ’14– (non-linear results),...
New system of coordinates

- It is more convenient to study the propagation in a different system of coordinates that somehow "follows" the collapse, denoted by $\hat{S}$.

\[ \hat{t} = t - F(r), \quad F(r) \sim -\frac{1}{2} \kappa \pm \ln |r - r \pm | \text{ for } r \text{ near } r \pm. \]

In $(\hat{t}, r, \omega)$ the metric is smooth across $r = r \pm$.

- We are studying the backward propagation. Nothing can cross the horizons. Hence the propagation takes place in compact smooth slices.

- After possibly rescaling, in $\hat{S}$ the collapsing star is given by $B = \{ (t, z(\hat{t}), \omega) \}$, $z(\hat{t}) = r - \alpha (\hat{t} - 1) + O(\hat{t} - 1)^2$. 
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Propagation in $\mathcal{S}$

$$\hat{t} \quad r = r_- \quad r = r_+$$

$$\hat{t} = T$$

$$\hat{t} = 1$$

$u_0, u_1$

$A, B, C$
Why study propagation in \( \hat{S} \) instead of \( S_\ast \)?

- Due to the blueshift the wave gets localized on a region of size

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▶ In $S_*$ the boundary affects the propagation for $t \in [0, T/2]$. A harder high frequency analysis is required: it needs to work for time intervals of size $T/2 \to \infty$.

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Backward scattering fields

▶ Goal: understand the behavior as $\hat{t} \to -\infty$ to solutions of $(\Box + m^2)u = 0$. 

The equation is invariant under time-reversion $t \mapsto -t$. It suffices to understand forwards scattering, then reverse time.

Under time reversion, the surface $\hat{t} = -\infty$ becomes $\{r = r -\} \cup \{r = r +\}$. Therefore: scattering fields are obtained by tracing forwards solutions along the horizons, then reversing time.

This constructs $u^+$ and $u^-$. Melrose–Sá-Barreto–Vasy '08 (later extended by Dyatlov '12 and Vasy '13) shows that they decay exponentially.

This strategy is due to Friedlander '80s (in the more complicated Euclidean scattering). For related perspectives in MGR, see Gérard–Georgescu–Häfner '14-'17, Nicolas '17, Dafermos–Rodnianski–Shlapentokh-Rothman '17.
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Melrose–Sá-Barreto–Vasy '08 (later extended by Dyatlov '12 and Vasy '13) shows that they decay exponentially.

This strategy is due to Friedlander '80s (in the more complicated Euclidean scattering). For related perspectives in MGR, see Gérard–Georgescu–Häfner '14-'17, Nicolas '17, Dafermos–Rodnianski–Shlapentokh-Rothman '17.
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**Theorem**

Let $u$ be a solution written in $\hat{S}$ of

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Then for some $\nu > 0$, 

$$v_\pm(x, \omega) = O(e^{-\nu x})$$ for large $x$. 

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$$u(\hat{t}, r, \omega) - (v_+ + v_-)(T - \hat{t} - 2F(r), \omega) = O(e^{-\nu T})$$ as $T \to +\infty$. 

Backward scattering fields
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Semiclassical description of the blueshift effect

Near the black holes, asymptotically backwards waves look like

\[ u_-(T - 2F(r), \omega) \]

where \( u_-(x, \omega) = 0 \) for \( x \leq 0 \) and decays exponentially for \( x \geq 0 \).

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- Using \( F(r) \sim -(2\kappa_-)^{-1} \ln(r - r_-) \) near \( r = r_- \),
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  u_-(T - 2F(r), \omega) \sim u_-(T + \frac{1}{\kappa_-} \ln(r - r_-), \omega) = u_-(\ln \left(\frac{r - r_-}{h}\right), \omega).
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The semiclassical wavefront set of the \( h \)-dependent distribution

\[ u_- \left( \ln \left( \frac{r - r_-}{h} \right), \omega \right) \]

satisfies \( \text{WF}_h \subset \{(r_-, \omega, \xi, 0)\} \). This gives a semiclassical description of the blueshift effect.
Study of the reflection

\[ r = r_- \]

\[ \hat{t} = 1 \]

\[ r = r_+ \]
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\[ \pi(WF_h(u)) \]

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- Essentially one reflection that occurs at $r = r_-, \hat{t} = 1$. 

As a consequence we can study the boundary problem near $r = r_-, \hat{t} = 1$. There the K–G operator is well approximated by a constant coefficients operator with symbol $\sigma(\Box g)(1, r-, 0; \tau, \xi, 0)$.

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This gives a good enough approximation of $u$ after reflection for times in $[1 - ch, 1]$ for any fixed $c > 0$. 
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Zoom in a box of size $O(h)$ near $r = r_-$ and $\hat{t} = 1$

$r = r_-$

$\sim h$

$\hat{t} = 1$

$\hat{t} = 1 - ch$

$\sim h$

$B$
Global study of the reflection

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- As the initial data is localized in frequencies $\sim h^{-1}$, we can construct a WKB approximate solution for $(\Box + m^2)u_{WB} = 0$. 

- By H"ormander's hyperbolic energy estimates, $u$ (the solution with boundary) is well approximated by this explicit WKB parametrix for $t \in [0, 1 - ch]$, with error of order $O(h) = O(e^{-\kappa - T})$. 

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- Going back to $\mathcal{S}_*$, we get the theorem:

**Theorem [D '17]**

If $u$ solves

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\begin{cases}
(\Box_g + m^2)u = 0 \\
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\end{cases}
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then there exist $u_-, u_+$ smooth and exponentially decaying; and $c_0 > 0$ such that for $t$ near 0, in $\mathcal{S}_*$

$$
u(0, x, \omega) = \frac{r_-}{r} u_-(\frac{1}{\kappa_-} \ln \left( \frac{x}{e^{-\kappa_- T}} \right), \omega) \quad \text{WKB part from BH}
$$

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+ u_+(T - x, \omega) \quad \text{scattering part to CH} + O_{\mathcal{H}^{1/2}} (e^{-c_0 T}).
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This describes the PDE part of the problem. A delicate calculation remains to derive Hawking’s radiation from here.
Extensions to non-symmetric backgrounds

The simplest class consists of metric of the form

\[ g = g_0 + \varepsilon \eta, \]

where \( g_0 \) is the SdS metric; \( \eta = \eta(r, \omega, dr, d\omega) \) is smooth and vanishes in neighborhoods of \( r_{\pm} \); and \( \varepsilon \) is small.
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Asymptotic of scalar fields

Theorem [work in progress]

Consider \( u_0, u_1 \) smooth with compact support, and \( u \) solution of

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There exist \( u^-, u^+ \) smooth and exponentially decaying with

\[
\begin{align*}
u(0, x, \omega) &= a(0, x, \omega) \\
&\times \left( \phi(0, x, \omega) - \ln(\phi(0, x, \omega)) e^{-\kappa - T} \right) + u^+(T-x, \omega) + O(H^{1/2}(e^{-c_0 T})).
\end{align*}
\]

where:

\( \phi \) solves the eikonal equation \( g(\nabla \phi, \nabla \phi) = 0 \);

\( \psi: \mathbb{R}^2 \times S^2 \to S^2 \) solves the linearized eikonal equation \( g(\nabla \phi, \nabla \psi) = 0 \);

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Thank you!