Practice Midterm II.

Each answer must be mathematically justified.

Problem 1. True or false? Justify your answer. (4 points each)
(a) There exists a function $\varphi : \mathbb{R} \to \mathbb{R}$ defined near $x = 0$ such that
$$\varphi(x)^4 - \varphi(x)^3 = x, \quad \varphi(0) = 1, \quad \varphi'(0) = 1.$$
(b) A convex function must have a global minimum.
(c) The following minimization problem has 9 as solution:
$$\min x^2 + 4xy + 4y^2 \quad \text{subject to } x \geq 1, \ y \geq 1$$
(d) The following set is compact:
$$\{(x, y) \in \mathbb{R}^2 : x^2 - 2xy + 2y^2 - 4y \leq -3\};$$
(e) Every function defined on a compact set must have a global minimum.

Problem 2. The two questions are independent. (10 points each)
(a) Show that
$$2e^{(a+b)^2} \leq e^{4a^2} + e^{4b^2}.$$
(b) Find the global minimum of the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by
$$g(x, y) = e^{x^2} + y^2 - 2xy + 2x^2.$$

Problem 3. (a) (5 points) Show that the set
$$K = \{(x, y) \in \mathbb{R}^2 : x^2 - 2x + y^2 - 2y \leq 3\}$$
is compact. (Hint: write $x^2 - 2x + y^2 - 2y$ as a sum of squares).
(b) (10 points) Solve the minimization problem
$$\min x + 2y \quad \text{subject to } x^2 - 2x + y^2 - 2y = 3.$$
(c) (5 points) Solve the minimization problem
$$\min x + 2y \quad \text{subject to } x^2 - 2x + y^2 - 2y \leq 3.$$

Problem 4. The two questions are independent. (10 points each)
(a) Define $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \frac{1}{1 + x^2} + e^{-x^2}.$$Show that for any $y \in (0, 2)$, the equation $f(x) = y$ has at least two solutions.
(b) Show that if $\varepsilon > 0$ is sufficiently small, the system
$$\begin{cases} x + y - \varepsilon x^3 = 2 \\ 4x + y = 5 \end{cases}$$has at least one solution.

Problem 5. (20 points) Solve the minimization problem
$$\min x^2 + 2y^2 + 2xy - 8x - 8y \quad \text{subject to } x \geq 0, \ y \geq 3.$$