Practice final

Each answer must be mathematically justified. Don’t forget your name.

Problem 1. True or False? Justify your answer.
(a) The solution of the minimization problem
\[ \min x^2 - 2xy + e^x \text{ subject to } x \geq 0, \ y \geq -2 \]
is 2.
(b) A matrix is diagonalizable if and only if it has distinct eigenvalues.
(c) Every convex function admits a global minimum.
(d) If \( U \) and \( V \) are two open subsets of \( \mathbb{R}^N \), then the set
\[ \{ x \in \mathbb{R}^N : x \in U \text{ and } x \in V \} \]
is open.
(e) The maximization problem
\[ \min I[y], \ y(0) = 3, \ y(\ln 2) = 3, \quad \text{where } I[y] = \int_0^{\ln 2} (y^2 - 2y' + y'^2)dt \]
has a solution, reached at \( y = e^t + 2e^{-t} \).

Problem 2. Solve the ODE
\[ -y'' - (2t - 3)y'^2 = 0, \ y'(0) = 1/2, \ y(3) = -\ln(2). \]
Check your solution.

Problem 3. Let
\[ f(x, y) = \frac{4 + \sin(x)}{1 + x^2 + y^2}. \]
(a) Show that the set
\[ K = \{(x, y) : x^2 + y^2 \leq 4\} \]
is compact.
(b) Deduce that the maximization problem
\[ \max f(x, y) \text{ subject to } (x, y) \in K \]
admits a solution with a corresponding maximum at least 4.
(c) Show that outside \( K \), \( f(x, y) > 1 \).
(d) Deduce that the maximization problem
\[ \max f(x, y) \text{ subject to } (x, y) \in \mathbb{R}^2 \]
admits a solution.
**Problem 4.** Let $T$ be a symmetric matrix. Find a necessary and sufficient condition on $T$ so that there exist a symmetric matrix $R$ such that $R^2 = T$.

**Problem 5.** Solve the maximization problem
\[ \max I[y] \text{ with } y(0) = 0, \quad \text{where } I[y] = \int_0^1 (24t^2y - \dot{y}^2)dt. \]

**Problem 6.** Show that for $\varepsilon$ sufficiently close to 0, the system of equations
\[
\begin{align*}
4x + y + \varepsilon ye^x &= 5 \\
2x - 3y + \varepsilon(y^3 + x^3) &= -1
\end{align*}
\]
has a unique solution $(x, y)$ close to $(1, 1)$.

**Problem 7.** The goal of this problem is to solve the minimization problem
\[(P): \min 6x + 3y - 3z - 1 \text{ subject to } (x - 1)^2 + (x + y - z)^2 + z^2 \leq 2\]
We set $f(x, y, z) = 6x + 3y - 3z - 1$ and
\[U = \{(x, y, z) : (x - 1)^2 + (x + y - z)^2 + z^2 < 2\}.
\]
(a) Show that $f$ has no minimum or maximum on $U$.
(b) Solve the minimization problem
\[\min 6x + 3y - 3z - 1 \text{ subject to } (x - 1)^2 + (x + y - z)^2 + z^2 = 2\]
(c) Solve the minimization problem (P).

**Problem 8.** Solve the maximization problem
\[ \max e^{x+y} \text{ subject to } x^2 + y^2 \leq 2. \]
(Hint: use convexity)

**Problem 9.** Solve the minimization problem
\[ \min x^2 + 2y^2 + 2xy - 8x - 8y \text{ subject to } x \geq 1, \ y \geq 2. \]

**Problem 10.** Solve the ODE
\[
\begin{aligned}
\dot{x} &= -2x - 2y - 6z \\
\dot{y} &= -2x + y \\
\dot{z} &= x + z
\end{aligned}
\]