Problem 1. (25 points) True or False?

- Report your answers in the table below.
- For this problem only, you do not have to justify your solutions.

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(a) (5 points) For every \(a \in \mathbb{R}\) and \(b \in \mathbb{R}\),

\[
\sqrt{1 + \left(\frac{a+2b}{3}\right)^2} \geq \frac{1}{3} \sqrt{1 + a^2} + \frac{2}{3} \sqrt{1 + b^2}
\]

False: take \(a = 0, b = 3\), get

\[
\sqrt{5^2} \text{ on the left,}
\]

\[
\frac{1}{3} \sqrt{1^2} + \frac{2}{3} \sqrt{10^2} \text{ on the right,}
\]

and \(\sqrt{5^2} - \frac{1}{3} \sqrt{1^2} - \frac{2}{3} \sqrt{10^2} < \sqrt{5} - \frac{1}{3} - \frac{2}{3} \sqrt{9} < \sqrt{5} - \frac{4}{3} < 0\)

Observe that this is negative because:

\[
5 < \frac{49}{9}. \quad (\Leftrightarrow 45 < 49).
\]
(b) (5 points) There exists a function $f$ defined near 0, such that $f(0) = 2$, $f'(0) = 1$ and
\[ f(x)^3 + f(x) = x + 10. \]
False. If $g$ existed we would have
\[ 3g'(x)g(x)^2 + g'(x) = 1 \]
\[ \Rightarrow \quad 3g'(0) \cdot 2^2 + g'(0) = 1 \]
\[ \Rightarrow \quad g'(0) = \frac{1}{13} \neq 1. \]

(c) (5 points) $5$ is the solution of the optimization problem
\[ \max 2x + y + z \quad \text{subject to} \quad x^2 + y^2 + z^2 \leq 6. \]
False: take $x=2$, $y=1$, $z=1$ and get
\[ 2x + y + z = 6 \]
\[ x^2 + y^2 + z^2 = 6 \leq 6. \]
(d) (5 points) If \( f : \mathbb{R}^2 \to \mathbb{R} \) is a smooth convex function then the function \( \varphi : \mathbb{R} \to \mathbb{R} \) given by \( \varphi(x) = f(x, -2x) \) is convex.

True. E.g., find \( \varphi''(x) \):

\[
\varphi'(x) = (3x^2 - 2xy)(x, -2x)
\]

\[
\varphi''(x) = (3x^2 - 2x \cdot 2y + 4y^2)(x, -2x)
\]

But \( g \) is convex, therefore

\[
\begin{bmatrix}
x^2 & xy
\end{bmatrix}
\begin{bmatrix}
x
y
\end{bmatrix} \geq 0 \Rightarrow \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix}
x^2 & xy
\end{bmatrix} \begin{bmatrix} x \\
y
\end{bmatrix} \geq 0
\]

\[
\Rightarrow 3x^2 - 2x \cdot 2y + 4y^2 \geq 0 \Rightarrow \varphi''(x) \geq 0.
\]

(There is another simpler method using definition of convex functions.)

(e) (5 points) If \( K \subseteq \mathbb{R}^n \) and \( L \subseteq \mathbb{R}^n \) are two compact sets, then the set

\[
M = \{ \vec{x} \in \mathbb{R}^n : \vec{x} \in K \text{ or } \vec{x} \in L \}
\]

is compact.

True. If \( K, L \) are compact, then they are both bounded:

\[
K \subseteq B(0, R) \Rightarrow M \subseteq B(0, \max(R, R'))
\]

\[
L \subseteq B(0, R')
\]

\[
\Rightarrow M \text{ is bounded.}
\]

\[
M \text{ is also closed}: \text{ if } \vec{x}_n \in M \text{ with } \vec{x}_n \to \vec{x}_\infty, \text{ then for each } \eta, \vec{x}_n \in K \text{ or } \vec{x}_n \in L.
\]

Since \( \vec{x}_n \) is an infinite sequence, there exist infinitely many \( n \) such that \( \vec{x}_n \in K \) or there exist infinitely many \( n \) such that \( \vec{x}_n \in L \). In the first case, since \( K \) is closed, \( \vec{x}_\infty \in K \). In the second case, since \( L \) is closed, \( \vec{x}_\infty \in L \). Thus \( \vec{x}_\infty \in M \Rightarrow M \text{ is closed.} \)
Problem 2. (25 points)
(a) (15 points) Show that for \( \varepsilon \) sufficiently close to 0, the function \( f : \mathbb{R}^2 \to \mathbb{R} \)
defined by
\[
f(x, y) = x^3 - 2xy + 2y^2 - 4y + \varepsilon e^x
\]
admits a critical point.

We write the critical point equations:
\[
\begin{align*}
\frac{\partial f}{\partial x}(x, y) &= 3x^2 - 2y + \varepsilon e^x = 0 \\
\frac{\partial f}{\partial y}(x, y) &= 2x - 4y + 2 - 4 = 0
\end{align*}
\]

Let \( F(\varepsilon; x, y) = \begin{bmatrix} 2x - 2y + \varepsilon e^x \\ -2x + 4y - 4 \end{bmatrix} \).

Note that:
(i) \( F(0; 2, 2) = 0 \)
(ii) \( \partial_{x,y} F(\varepsilon; x, y) = \begin{bmatrix} 2 + \varepsilon e^x & -2 \\ -2 & 4 \end{bmatrix} \)

\( \Rightarrow \partial_{x,y} F(0; 2, 2) = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \), \( \text{det} = 4 \neq 0 \)

\( \Rightarrow \partial_{x,y} F(0; 2, 2) \) is invertible.

Thus by the implicit function theorem, there exists a solution \( (x(\varepsilon), y(\varepsilon)) \)

\( F[\varepsilon; x, y] = [0] \) as long as \( \varepsilon \) is small.

Because of the definition of \( F \), \( (x(\varepsilon), y(\varepsilon)) \)
is a critical point of \( F \).
(b) (10 points) Show that for $\epsilon > 0$ sufficiently small, this critical point must correspond to a global minimum.

We show that $f$ is convex for $\epsilon > 0$:

$$f''(x,y) = \begin{bmatrix} 2 + 3e^x & -2 \\ -2 & 4 \end{bmatrix}$$

Note: $2 + 3e^x > 0$

$$4(2 + 3e^x) - 4 = 4 + 4\epsilon e^x > 0$$

when $\epsilon > 0$

$\Rightarrow f$ is convex.

When $0 < \epsilon$ is small, $f$ has a critical point. Therefore it must correspond to a global minimum (property of convex functions).
Problem 3. (25 points)
(a) (5 points) Show that the set $K$ given by
\[ \{(x, y) \in \mathbb{R}^2 : (x + y)^2 + x^2 \leq 1\} \]
is compact.

We first show that $K$ is closed.
If $(x_n, y_n) \in K$ and $(x_n, y_n) \to (x_\infty, y_\infty) \in \mathbb{R}^2$
then $(x_n + y_n)^2 + x_n^2 \leq 1 \Rightarrow (x_\infty + y_\infty)^2 + x_\infty^2 \leq 1$
by passing to limit preserves non-
strict inequalities.
\[ \Rightarrow (x_\infty, y_\infty) \in K \Rightarrow K \text{ is closed.} \]

We now show that $K$ is bounded.
If $(x, y) \in K$, $(x + y)^2 + x^2 \leq 1$
\[ \Rightarrow \begin{cases} (x + y)^2 \leq 1 \\ x^2 \leq 1 \end{cases} \]
\[ \Rightarrow \begin{cases} -1 \leq x + y \leq 1 \\ -1 \leq x \leq 1 \end{cases} \Rightarrow \begin{cases} -2 \leq y \leq 2 \\ -1 \leq x \leq 1 \end{cases} \]
\[ \Rightarrow \begin{cases} y^2 \leq 4 \\ x^2 \leq 1 \end{cases} \Rightarrow x^2 + y^2 < 9 \Rightarrow K \subset B(0,0; 3). \]
\[ \Rightarrow K \text{ is bounded.} \]
\[ \Rightarrow K \text{ is compact.} \]
(b) (10 points) Solve the maximization problem

\[
\max 2x^2 - 4x + y^2 + 2xy \text{ subject to } (x+y)^2 + x^2 = 1.
\]

The Lagrangian is \( \mathcal{L}(x,y; \lambda) = 2x^2 - 4x + y^2 + 2xy - \lambda ((x+y)^2 + x^2 - 1) \).

The critical point equations are

\[
\begin{align*}
4x - 4 + 2y - 2\lambda (x+y + x) &= 0, \\
2y + 2x - 2\lambda (x+y) &= 0, \\
(x+y)^2 + x^2 - 1 &= 0
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
(2x+y)(1-2\lambda) = 0 \\
(x+y)(1-\lambda) = 0 \\
(x+y)^2 + x^2 - 1 = 0
\end{cases}
\Rightarrow x+y = 0 \text{ or } \lambda = 1.
\]

If \( \lambda = 1 \), then \( 2x + y = -2 \Rightarrow x + y = -2 - x \)

\[
\Rightarrow (2-2x)^2 + x^2 - 1 = 0
\]

\[
\Rightarrow 2x^2 + 4x + 3 = 0. \quad \Delta = 16 - 4 \cdot 3 \cdot 2 = -8, \text{ no solutions.}
\]

Hence \( x+y = 0 \Rightarrow x^2 = 1 \Rightarrow \{y = -1 \text{ or } y = 1\}. \)

The first case achieves the value -3 and the second 5.

Hence the only candidate is \((-1,1)\) with value 5.

Hence \( \lambda = \frac{3}{2} \) and we must check that

\[
\mathcal{L}(x,y; -\frac{3}{2}) = 2x^2 - 4x + y^2 + 2xy + \frac{3}{2} ((x+y)^2 + x^2 - 1)
\]

is concave:

\[
\mathcal{L}''(x,y; -\frac{3}{2}) = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}
\]

1st minor = -2, 2nd minor = 1 \Rightarrow \boxed{\text{max is 5}}
(c) (10 points) Solve the maximization problem

$$\max 2x^2 - 4x + y^2 + 2xy \text{ subject to } (x+y)^2 + x^2 \leq 1.$$ 

1. A maximum must exist: we are maximizing the continuous function

$$f(x,y) = 2x^2 - 4x + y^2 + 2xy$$
over the compact set $K$

(extreme value theorem)

2. It cannot be reached on

$$U = \{ (x,y) \in \mathbb{R}^2 : (x+y)^2 + x^2 \leq 1 \}.$$ 
Indeed, $U$ is open:

$$U = \{ (x,y) \in \mathbb{R}^2 : (x+y)^2 + x^2 > 1 \}$$
is closed because if $(x_n, y_n) \in \overline{U}$, with $(x_n, y_n) \to (x_\infty, y_\infty)$ then

$$(x_n + y_n)^2 + x_n^2 \geq 1$$
$$\Rightarrow (x_\infty + y_\infty)^2 + x_\infty^2 \geq 1 \Rightarrow (x_\infty, y_\infty) \geq 1.$$ 

In addition $f$ has no critical points in $U$: 

$$\begin{cases} 4x - 4 + 2y = 0 \\ 2y + 2x = 0 \end{cases} \Rightarrow \begin{cases} x = 2 \\ x + y = 0 \Rightarrow y = -2 \end{cases}$$

and $(2,-2) \notin U$: $(2-2)^2 + 2^2 = 4 > 1$.

3. Therefore it must be attained at a point

with $(x+y)^2 + x^2 = 1$. (b) shows that

it must be 5.
Problem 4. (25 points) Solve the minimization problem
\[ \min x^2 + 2y^2 + 2xy - 8x - 8y \quad \text{subject to} \quad x \geq 1, \quad y \geq 2. \]

**Introduce Kuhn-Tucker conditions:**
\[
\min x^2 + 2y^2 + 2xy - 8x - 8y \quad \text{subject to} \quad -x \leq -1, \quad -y \leq -2
\]
\[
L(x,y; \lambda, \mu) = x^2 + 2y^2 + 2xy - 8x - 8y - \lambda(-x+1) - \mu(-y+2)
\]
\[
= x^2 + 2y^2 + 2xy - 8x - 8y + \lambda(x-1) + \mu(y-2).
\]

**Critical point equations:**
\[
\begin{align*}
2x + 2y - 8 + \lambda &= 0 \\
4y + 2x - 8 + \mu &= 0 \\
\lambda &\leq 0 \quad \text{with} \quad x > 1 \implies \lambda = 0 \\
\mu &< 0 \quad \text{with} \quad y > 2 \implies \mu = 0.
\end{align*}
\]

**Case I:** Both constraints are active: \(x=1, y=2\).

Get value -11.

**Case II:** First constraint is active: \(x=1\)

Second constraint is inactive: \(y > 2 \implies \mu = 0\).

\[
\begin{align*}
2 + 2y - 8 + \lambda &= 0 \\
4y + 2 - 8 &= 0 \implies y = 3/2 < 2, \text{ no solution.}
\end{align*}
\]

\[
\begin{align*}
\lambda &\leq 0, \quad y > 2 \quad \text{**}
\end{align*}
\]
Case III: First constraint is passive: \( x > 1, \lambda = 0 \)
Second constraint is active: \( y = 2 \)

\[
\begin{align*}
2x + 4 - 8 &= 0 \\
8 + 2x - 8 + \mu &= 0 \quad \Rightarrow \quad \mu = -4 \\
\lambda &= 0, x > 1 \\
\mu &\leq 0, y = 2
\end{align*}
\]

Get value -12. \((2, 2)\) is better candidate than \((1, 2)\).

Case IV: Both constraints are passive: \( x > 1, \lambda = 0, y > 2, \mu = 0 \)

Get:

\[
\begin{align*}
2x + 2y - 8 &= 0 \\
4y + 2x - 8 &= 0 \quad \Rightarrow \quad y = 0 \\
\lambda = \mu &= 0, x > 1, y > 2 \\
\ast &\quad \Rightarrow \text{no solutions.}
\end{align*}
\]

To conclude: we must check that

\( L(x, y; 0, 2) \) is convex.

\[
L(x, y; 0, 2) = x^2 + 2y^2 + 2xy - 8x - 8y + 2(y-2)
\]

\[
= x^2 + 2y^2 + 2xy - 8x - 6y - 4.
\]

\[
L''(x, y; 0, 2) = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}; \quad 2 > 0
\]

\[
\Rightarrow \quad L''(x, y; 0, 2) > 0 \Rightarrow L \text{ is convex} \Rightarrow -12 \text{ is the solution.}
\]