Analysis and Optimization,
Midterm 2 (75 minutes)

Name: 
Uni:

- Write your name above AND on page 2.
- This exam booklet contains 16 pages and 4 problems, each graded out of 25 points.
- You may not use your notes, books, phones and calculating devices.
- If you need more space, use the extra sheets at the back. Indicate clearly that you did so.
<table>
<thead>
<tr>
<th>Problem</th>
<th>Grade</th>
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Problem 1. (25 points) True or False?

- Report your answers in the table below.
- **For this problem only,** you do not have to justify your solutions.

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<tr>
<th>Question</th>
<th>True/False?</th>
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(a) (5 points) For every $a \in \mathbb{R}$ and $b \in \mathbb{R}$, 

$$\sqrt{1 + \left( \frac{a + 2b}{3} \right)^2} \geq \frac{1}{3} \sqrt{1 + a^2} + \frac{2}{3} \sqrt{1 + b^2}$$
(b) (5 points) There exists a function $f$ defined near 0, such that $f(0) = 2$, $f'(0) = 1$ and
\[ f(x)^3 + f(x) = x + 10. \]

(c) (5 points) 5 is the solution of the optimization problem
\[
\max 2x + y + z \quad \text{subject to} \quad x^2 + y^2 + z^2 \leq 6.
\]
(d) (5 points) If \( f : \mathbb{R}^2 \to \mathbb{R} \) is a smooth convex function then the function \( \varphi : \mathbb{R} \to \mathbb{R} \) given by \( \varphi(x) = f(x, -2x) \) is convex.

(e) (5 points) If \( K \subset \mathbb{R}^n \) and \( L \subset \mathbb{R}^n \) are two compact sets, then the set
\[
M = \{ \overrightarrow{x} \in \mathbb{R}^n : \overrightarrow{x} \in K \text{ or } \overrightarrow{x} \in L \}
\]
is compact.
Problem 2. (25 points)
(a) (15 points) Show that for $\varepsilon$ sufficiently close to 0, the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by
\[ f(x, y) = x^2 - 2xy + 2y^2 - 4y + \varepsilon e^x \]
admits a critical point.
(Intentionally left blank)
(b) (10 points) Show that for $\varepsilon > 0$ sufficiently small, this critical point must correspond to a global minimum of $f$. (You can use the result of 2(a) to solve this question, even if you did not solve 2(a)).
Problem 3. (25 points)
(a) (5 points) Show that the set $K$ given by
\[ K = \{(x, y) \in \mathbb{R}^2 : (x + y)^2 + x^2 \leq 1\} \]
is compact.
(b) (10 points) Solve the maximization problem

$$\max 2x^2 - 4x + y^2 + 2xy \quad \text{subject to} \quad (x + y)^2 + x^2 = 1.$$
(c) (10 points) Solve the maximization problem
\[ \max \ 2x^2 - 4x + y^2 + 2xy \ \text{subject to} \ (x + y)^2 + x^2 \leq 1. \]
(You can use the result of 3(a) to solve this question, even if you did not solve 3(a))
Problem 4. (25 points) Solve the minimization problem
\[ \min x^2 + 2y^2 + 2xy - 8x - 8y \] subject to \( x \geq 1, \; y \geq 2. \)
(Additional page)
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