Introduction to modern analysis I, Midterm 1 (75 minutes)

Name:
Uni:

• Write your name above AND on page 2.
• This exam booklet contains 12 pages and 5 problems, each graded out of 20 points.
• You may use one single, one-sided paper sheet of notes
• No books, phones and calculating devices.
• If you need more space, use the extra sheets at the back. Indicate clearly that you did so.
Name:
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Problem 1. (20 points) True or False?

- Report your answers in the table below.
- For this problem only, you do not have to justify your solutions.
- I am not taking points off for wrong answers, so make sure to fill every box with a True/False.

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(a) (4 points) The set

\[ E = \left\{ \frac{1}{1 + n^2} + (n + m)^2 : n \in \mathbb{N}, m \in \mathbb{N} \right\} \subset \mathbb{R} \]

has no accumulation point.
(b) (4 points) Every bounded sequence is convergent.

(c) (4 points) Let $A \subset \mathbb{R}$, $B \subset \mathbb{R}$. Define

$$C = \{ab : a \in A, b \in B\}.$$ 

If $A$ and $B$ are countable, then $C$ is countable.
(d) (4 points) The set

\[ [0, 1) \times (0, 1) \]

is neither an open nor a closed subset of \( \mathbb{R}^2 \).

(e) (4 points) If \( X \) is a metric space and \( U \subset X \) is an open set then every accumulation point of \( U \) must be outside \( U \).
Problem 2. (20 points)

Let \( \{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R} \) be a sequence. Assume that for every \( n \in \mathbb{N} \), \( x_n < 1 \) and that \( x_n \) converges to 1. Define

\[ E = \{x_n : n \in \mathbb{N}\} \]

Prove that \( \sup E \) exists and is equal to 1.
Problem 3. (20 points)
Let $X$ be a metric space and $y_0, y_1 \in X$. Show that the set

$$\{x \in X : d(x, y_0) + d(x, y_1) \leq 1\}$$

is closed.
Problem 4. (20 points)
Assume that \( \{x_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^2 \) is a bounded sequence. Define
\[
E = \{x_n : n \in \mathbb{N}\}
\]
Prove that if \( E \) has a single accumulation point then \( x_n \) is convergent.
Problem 5. (20 points)
Let $X$ be a metric space, $K \subset X$ be a compact space and $f : X \to \mathbb{R}$. Assume that for every $x \in K$, there exists $r_x > 0$ and $a_x > 0$ such that
\[ y \in N_{r_x}(x) \Rightarrow f(y) \geq a_x. \]
Show that there exists $A > 0$ such that for every $z \in K$, $f(z) \geq A$.
Would this conclusion hold if $K$ was not compact?
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