HW4 - Due 02/20

Each answer must be mathematically justified. Don’t forget your name.

Problem 1. Are the following sets convex?
(a) \( A = \{ \vec{x} \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 \leq 1 \} \);
(b) \( B = \{ \vec{x} \in \mathbb{R}^n : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, \ldots, 0 \leq x_n \leq 1 \} \);
(c) \( C = \{ (x, y) \in \mathbb{R}^2 : y \geq x^3 - x \} \);
(d) \( D = \{ (x, y, z) \in \mathbb{R}^3 : z = 2x^2 + y^2 \} \).

Problem 2. Prove the following inequalities:
(a) For any \( a, b \) real numbers, \( 3e^{a+2b} \leq e^{3a} + 2e^{3b} \);
(b) For any \( a, b \) real numbers, \( (a + b)^{2018} \leq 2^{2017}a^{2018} + 2^{2017}b^{2018} \).

Problem 3. Show that if \( u(t) \) is convex and increasing and \( f(\vec{x}) \) is convex, then the function \( g(\vec{x}) = u(f(\vec{x})) \) is convex. Use this fact to prove that the function
\[
\begin{align*}
h(x, y) &= e^{x^2+y^4+1}
\end{align*}
\]
is convex.

Problem 4. Find the global minimum of the following functions:
(a) \( f(x, y) = e^{x^2+2xy+4y^2} \) over all real \( x \) and \( y \);
(b) \( g(x, y) = 2x^2 - 2x + 1 - 2xy + y^2 \) over \( \{(x, y) : (x - 1)^2 + (y - 1)^2 \leq 1 \} \);
(c) \( h(x, y) = x^2 + y^2 + 1 \) over \( \{(x, y) : 1 \leq x \leq 2, 1 \leq y \leq 2 \} \).
(d) (optional) \( i(x, y) = e^{x^2+y^2} + e^{-x^2-y^2} \) over all real \( x \) and \( y \).

Problem 5. For which values of \( a \) and \( b \) is the function
\[
\begin{align*}
f(x, y) &= e^{ax^2+by^2}
\end{align*}
\]
convex?

Problem 6. Show that if \( f \) is a convex smooth function on \( \mathbb{R} \), then for any value of \( a \), the graph of \( f \) is above the tangent line at \( a \). (Hint: use the fundamental theorem of calculus).