HW6 - Due 03/06

Each answer must be mathematically justified. Don’t forget your name.

Problem 1. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by
\[
f(x, y) = (e^x + 2e^y + x^2 + y^2)^2 - x^2 - y^2 - 10, \quad (x, y) \in \mathbb{R}^2.
\]
The goal of this problem is to show that \( f \) has a global minimum on \( \mathbb{R}^2 \), without explicitly finding it (this is impossible to do algebraically). We set
\[
K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 10\}.
\]

(a) Show that \( K \) is compact.
(b) Show that \( f \) restricted to \( K \) (i.e. the function \( f : K \to \mathbb{R} \)) admits a global minimum, and that this minimum is nonpositive. (Hint: you are not required to find this minimum).
(c) Show that \( f(x, y) > 0 \) for every \( (x, y) \) outside \( K \).
(d) Deduce that \( f \) has a global minimum on \( \mathbb{R}^2 \), which is moreover attained in \( K \).

(a) To show that \( K \) is compact, it suffices to show that it is both bounded and closed. We observe that
\[
x^2 + y^2 \leq 10 \Rightarrow x^2 + y^2 < 16.
\]
Therefore \( K \) is bounded. We now observe that \( K \) is closed: assume that \( (x_n, y_n) \) is a sequence of points in \( K \), which converges to a point \( (x_\infty, y_\infty) \in \mathbb{R}^2 \). Then,
\[
x_n^2 + y_n^2 \leq 10
\]
which goes to the limit as \( x_\infty^2 + y_\infty^2 \leq 10 \). Therefore, \((x_\infty, y_\infty) \in K \), and \( K \) is closed. We deduce that \( K \) is compact.
(b) Since \( K \) is compact, \( f \) restricted to \( K \) has a global minimum (in \( K \)). Since \((0, 0) \in K \) and \( f(0, 0) = -1 \), this minimum must be smaller or equal to \(-1 \), hence nonpositive.
(c) Observe that for any \( x, y \),
\[
e^x + 2e^y + x^2 + y^2 > x^2 + y^2 \geq 0 \Rightarrow (e^x + 2e^y + x^2 + y^2)^2 > (x^2 + y^2)^2.
\]
Now if \((x, y)\) does not belong to \( K \) then
\[
x^2 + y^2 > 10 \Rightarrow (x^2 + y^2)^2 > 10(x^2 + y^2) \Rightarrow (x^2 + y^2)^2 - x^2 - y^2 > 9(x^2 + y^2) > 90.
\]
This implies
\[
(e^x + 2e^y + x^2 + y^2)^2 - x^2 - y^2 - 10 > (x^2 + y^2)^2 - x^2 - y^2 - 10 > 90 - 10 = 80.
\]
Thus \( f > 0 \) outside \( K \).
(d) Let \( m \) be the global minimum of \( f \) in \( K \); note that \( m \) exists because of (b), and that \( m \leq 0 \). Then,
\[
\overrightarrow{x} \in K \Rightarrow f(\overrightarrow{x}) \geq m; \quad \overrightarrow{x} \notin K \Rightarrow f(\overrightarrow{x}) > 0 \geq m.
\]
Thus $f(\mathbf{x}) \geq m$ for every $\mathbf{x}$ in $\mathbb{R}^2$: this means that $m$ is the global minimum of $f$ in $\mathbb{R}^2$. Moreover because of (b) it is attained in $K$.

**Problem 2.** Set

$$A = \{(x, y) \in \mathbb{R}^2 : x \geq 1, y > 2\}.$$

(a) What is the closure of $A$?

(b) What is the interior of $A$?

(c) What is the boundary of $A$?

Provide mathematical proofs (a drawing is not a proof but can guide your intuition).

(a) We claim that the closure of $A$ is the set

$$C = \{(x, y) \in \mathbb{R}^2 : x \geq 1, y \geq 2\}.$$

Indeed, if $(x_\infty, y_\infty) \in \mathbb{A}$, then there exists a sequence $(x_n, y_n) \in \mathbb{A}$ with $(x_n, y_n) \to (x_\infty, y_\infty)$. Since $(x_n, y_n) \in A$,

$$x_n \geq 1, y_n > 2.$$

Passing to the limit transforms strict inequalities in non-strict inequalities (and does modify non-strict inequalities), thus

$$x_\infty \geq 1, y_\infty > 2.$$

This shows that $(x_\infty, y_\infty) \in C$, thus $\overline{A} \subset C$. To show the other inclusion, observe that every point in $C$ is the limit of points in $A$. Indeed, if $(x, y) \in C$, then

$$x \geq 1, y \geq 2 \implies x \geq 1; y + \frac{1}{n} > 2.$$

Therefore the sequence

$$(x_n, y_n) = \left(x, y + \frac{1}{n}\right)$$

takes values in $A$. Furthermore it converges to $(x, y)$. This proves $(x, y) \in \mathbb{A}$. Thus $C \subset \overline{A}$ hence $C = \overline{A}$.

(b) We prove that the interior of $A$ is the set

$$I = \{(x, y) \in \mathbb{R}^2 : x > 1, y > 2\}.$$

To prove that $I \subset \text{int}(A)$ we first show that every point in $I$ is the center of a ball contained in $A$. Fix $(x, y) \in I$. Then $x > 1$ and $y > 2$. Let $r = \min(x - 1, y - 2)$ (note that $r > 0$), and let us show that $B(x, y; r) \subset A$: if $(x', y') \in B(x, y; r)$ then by definition of $r$,

$$x' = x' - x + x \geq -|x' - x| + x \geq -\|(x', y') - (x, y)\| + x > -r + x > -(x - 1) - x > 1,$$

$$y' = y' - y + y \geq -|y' - y| + y \geq -\|(x', y') - (x, y)\| + y > -r + y > -(y - 2) - y > 2.$$

Thus $B(x, y; r) \subset A$. It follows that every point in $I$ is the center of a ball contained in $A$. Therefore $I \subset \text{int}(A)$. To show the other inclusion, we fix $(x, y) \in A$ a point
that is the center of a ball contained in $A$, and we show that $(x, y) \in I$. Let $B(x, y; r)$ be this ball. This ball contains the point

$$(x - \frac{r}{2}, y)$$

hence this point must be in $A$. It shows that

$$x - \frac{r}{2} \geq 1, \quad y > 2 \Rightarrow x \geq 1 + \frac{r}{2} > 1, \quad y > 2 \Rightarrow x > 1, \quad y > 2.$$ 

Thus $(x, y) \in I$. It follows that $\text{int}(A) \subset I$, hence $\text{int}(A) = I$.

(c) By definition,

$$\partial A = \overline{A} \setminus \text{int}(A) = \{(x, y) \in \mathbb{R}^2 : x \geq 1, \ y \geq 2\} \setminus \{(x, y) \in \mathbb{R}^2 : x > 1, \ y > 2\}$$

$$= \{(x, y) \in \mathbb{R}^2 : x = 1, \ y = 2\}.$$ 

Problem 3. Let $f : K \to \mathbb{R}$ be defined by

$$f(x, y) = x^2 - 4y^2, \quad (x, y) \in K, \quad K = \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 \leq 4\}.$$ 

The goal is to find the global minimum and maximum of $f$ in $K$.

(a) Show that $K$ is compact.

(b) Show that $f$ reaches its global minimum and maximum in $K$.

(c) Show that $f$ does not have local extrema in $\text{int}(K)$. Deduce that the global extrema of $f$ must be attained on $\partial K$.

(d) Show that $\partial K = \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 = 4\}$

(e) Show that $\partial K$ can be parametrized as follows:

$$\partial K = \{(2 \cos(t), \sin(t)) : t \in [0, 2\pi]\}.$$ 

(f) Find the global minimum and maximum of $f$ in $\partial K$, then find the global minimum and maximum of $f$ in $K$.

(a) To show that $K$ is compact, it suffices to show that it is both bounded and closed. We observe that $K \subset B(0, 0; 3)$ because

$$x^2 + 4y^2 \leq 4 \Rightarrow x^2 + y^2 \leq 4 \Rightarrow x^2 + y^2 < 9.$$ 

Therefore $K$ is bounded. We now observe that $K$ is closed: assume that $(x_n, y_n)$ is a sequence of points in $K$, which converges to a point $(x_\infty, y_\infty) \in \mathbb{R}^2$. Then,

$$x_n^2 + 4y_n^2 \leq 4$$

which transfers to the limit as $x_\infty^2 + 4y_\infty^2 \leq 4$. Therefore, $(x_\infty, y_\infty) \in K$, and $K$ is closed. We deduce that $K$ is compact.

(b) $f$ is continuous and $K$ is compact, hence $f$ reaches its global minimum and maximum in $K$.

(c) If $f$ had a global minimum or maximum in the interior of $K$, then they would be attained at a critical point, and the Hessian would be positive semi-definite or negative
semi-definite. That’s because the interior of $K$ is an open set. We now look for critical points of $f$:

$$\partial_x f(x, y) = 2x, \quad \partial_y f(x, y) = 8y, \quad f''(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & -8 \end{bmatrix}.$$ 

There is a single critical point $(0, 0)$; but the Hessian at this point is indefinite, therefore it corresponds to a saddle point. This shows that $f$ restricted to $\text{int}(K)$ cannot reach a minimum. Since $f$ reaches a global minimum on $K$, since this minimum cannot be attained in $\text{int}(K)$, its must be attained in $K \setminus \text{int}(K)$, i.e. in $\partial K$.

(d) We find $\partial K$. First, we show that

$$\text{int}(K) = \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 < 4\}.$$ 

It suffices to show that every point $(x, y)$ such that $x^2 + 4y^2 < 4$ is the center of a ball fully contained in $K$; and that every point $(x, y)$ such that $x^2 + y^2 = 4$ cannot be the center of a ball contained in $K$. We start with the former: if $x^2 + 4y^2 < 4$, let $0 < r < \frac{1}{2}(2 - \sqrt{x^2 + 4y^2})$ (note that such a value of $r$ exists because $x^2 + 4y^2 < 4$), and let us show that

$$B(x, y; r) \subset K.$$ 

If $(x', y') \in B(x, y; r)$ then

$$\sqrt{x'^2 + 4y'^2} = \|(x', 2y')\| \leq \|(x' - x, 2y' - 2y)\| + \|(x, 2y)\| \leq |x' - x| + 2|y' - y| + \sqrt{x^2 + 4y^2}.$$ 

Now, $|x' - x| < r$; and $|y' - y| < r$; thus by definition of $r$,

$$\sqrt{x'^2 + 4y'^2} < 3r + \sqrt{x^2 + 4y^2} < 2.$$ 

We deduce that $(x', y') \in K$, thus $B(x, y; r) \subset K$. This proves that

$$\text{int}(K) \subset \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 < 4\}.$$ 

To show the other inclusion, we must prove that every point $(x, y)$ such that $x^2 + y^2 = 4$ cannot be the center of a ball contained in $K$. Let $(x, y)$ with $x^2 + y^2$, and consider the ball $B(x, y; r)$. Consider the point

$$(x + r/2, y) \text{ if } x > 0; \quad (x - r/2, y) \text{ if } x < 0;$$

and observe that in both cases this point belong to $B(x, y; r)$. However, in either case it cannot belong to $K$. Indeed, if $x > 0$,

$$\left(x + \frac{r}{2}\right)^2 + 4y^2 = x^2 + 4y^2 + rx + \frac{r^2}{4} = 4 + rx + \frac{r^2}{4} > 4;$$

and if $x < 0$,

$$\left(x - \frac{r}{2}\right)^2 + 4y^2 = x^2 + 4y^2 - rx + \frac{r^2}{4} = 4 - rx + \frac{r^2}{4} > 4.$$ 

It follows that $\text{int}(K) = \{(x, y) : x^2 + y^2 < 4\}$ and since $K$ is closed,

$$\partial K = K \setminus \text{int}(K) = K \setminus \text{int}(K) = \{(x, y) : x^2 + 4y^2 \leq 4\} \setminus \{(x, y) : x^2 + 4y^2 < 4\}$$

$$= \{(x, y) : x^2 + 4y^2 = 4\}.$$
(e) If \( x^2 + 4y^2 = 4 \), then
\[
\left( \frac{x}{2} \right)^2 + y^2 = 1
\]
Thus
\[
\frac{x}{2} = \cos(t), \quad y = \sin(t).
\]
It follows that \( \partial K \) can be parametrized as announced above.

(f) We have
\[
f(2\cos(t), \sin(t)) = 4\cos^2(t) - 4\sin^2(t) = 4\cos(2t).
\]
We optimize \( 4\cos(2t) \): the minimum is \(-4\), attained at \( t = \pi/2 \) while the maximum if \( 4 \), attained at \( t = 0 \). Thus the minimum of \( f \) on \( \partial K \) is \(-4\), attained at \( (0, 1) \) while the maximum is \( 4 \), attained at \( (2, 0) \). Since \( f \) cannot have extrema in \( \text{int}(K) \) because of (c), these must be the extrema of \( f \) in \( K \).

**Problem 4.** (a) Solve the maximization problem
\[
\max 2x + y - z \quad \text{subject to} \quad 4x^2 + y^2 + z^2 = 12.
\]
(b) Solve the minimization problem
\[
\min 2x + y - z \quad \text{subject to} \quad 4x^2 + y^2 + z^2 = 12.
\]
(a) Define the Lagrangian
\[
\mathcal{L}(x, y, z; \lambda) = 2x + y - z - \lambda(4x^2 + y^2 + z^2 - 12).
\]
The necessary first order conditions for optimality are
\[
\begin{align*}
2 - 8\lambda x &= 0 \\
1 - 2\lambda y &= 0 \\
-1 - 2\lambda z &= 0 \\
4x^2 + y^2 + z^2 &= 12
\end{align*}
\]
\[
\Rightarrow \begin{align*}
x &= 1/(4\lambda) \\
y &= 1/(2\lambda) \\
z &= -1/(2\lambda)
\end{align*}
\]
\[
\Rightarrow \begin{align*}
x &= 1/(4\lambda) \\
y &= 1/(2\lambda) \\
z &= -1/(2\lambda)
\end{align*}
\]
\[
\Rightarrow \begin{align*}
x &= 1/(4\lambda) \\
y &= 1/(2\lambda) \\
z &= -1/(2\lambda)
\end{align*}
\]
\[
\Rightarrow \begin{align*}
\lambda &= \pm 1/4.
\end{align*}
\]
If \( \lambda = 1/4 \), then the solution is \( (x, y, z) = (1, 2, -2) \). We now observe that when \( \lambda = 1/4 \), the Lagrangian is
\[
\mathcal{L}(x, y, z; 1/4) = 2x + y - z + 3 - x^2 - \frac{y^2 + z^2}{4}.
\]
Its Hessian is
\[
\mathcal{L}''(x, y, z; 1/4) = \begin{bmatrix}
2 & 0 & 0 \\
0 & 1/2 & 0 \\
0 & 0 & 1/2
\end{bmatrix}
\]
which is negative definite, therefore \( \mathcal{L}(x, y, z; 1/4) \) is concave. The domain is \( \mathbb{R}^3 \), which is convex. It follows that \( (1, 2, -2) \) solves the maximization problem; the maximum is 6.
(b) If $\lambda = -1/4$, then the solution is $(x, y, z) = (-1, -2, 2)$. We now observe that when $\lambda = -1/4$, the Lagrangian is
\[
\mathcal{L}(x, y, z; -1/4) = 2x + y - z + 3 + x^2 + \frac{y^2 + z^2}{4}.
\]
Its Hessian is
\[
\mathcal{L}''(x, y, z; 1/4) = \begin{bmatrix}
2 & 0 & 0 \\
0 & 1/2 & 0 \\
0 & 0 & 1/2 \\
\end{bmatrix}
\]
which is positive definite, therefore $\mathcal{L}(x, y, z; 1/4)$ is convex. It follows that $(-1, -2, 2)$ solves the minimization problem; the minimum is $-6$. 