HW5 - Due 02/27

Each answer must be mathematically justified. Don’t forget your name.

Problem 1. Which of the sets below are closed? open? Justify.

(a) \( A = \{(x, y) \in \mathbb{R}^2 : x > 1; y > 1\} \);
(b) \( B = \{(x, y) \in \mathbb{R}^2 : \cos(x) + e^{xy} \geq 1\} \);
(c) \( C = \{(x, y, z) \in \mathbb{R}^3 : z^2 > 0; y \geq 1\} \);
(d) \( D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y - z = 1\} \).

(a) \ We want to show that \( A \) is open. Fix a point \((x_0, y_0) \in A\). Let \(0 < r_0 < \min(x_0 - 1, y_0 - 1)\) (note that \(r_0\) exists because \(x_0 > 1\) and \(y_0 > 1\)). We want to show that the ball \( B(x_0, y_0; r_0) \) is fully contained in \( A \). For that, fix \((x, y) \in B(x_0, y_0; r_0)\). Then
\[
|x - x_0| < r_0 \text{ and } |y - y_0| < r_0 \implies (x, y) \in A.
\]

(b) \ We want to show that \( B \) is closed. Fix a sequence \((x_n, y_n) \in B\), which converges to a point \((x_\infty, y_\infty) \in \mathbb{R}^2\): we must prove that \((x_\infty, y_\infty) \in B\). Since \((x_n, y_n) \in B\),
\[
\cos(x_n) + e^{x_n y_n} \geq 1;
\]

since \(\cos(x) + e^{xy}\) is the expression of a continuous function, the above equation still holds when passing to the limit \(n \to \infty\): therefore \(\cos(x_\infty) + e^{x_\infty y_\infty} \geq 1\). This means that \((x_\infty, y_\infty) \in B\). Thus \( B \) is closed.

(c) Observe that \((0, 1, 1/n) \in C\) for any \(n\). But, the limit as \(n \to \infty\) is \((0, 1, 0)\), which does not belong to \( C\): the condition \(z^2 > 0\) is not satisfied. Thus \( C \) cannot be closed. Since
\[
C = \{(x, y, z) \in \mathbb{R}^3 : z^2 > 0, y \geq 1\} = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0, y \geq 1\},
\]

the complement of \( C \) is
\[
\overline{C} = \{(x, y, z) \in \mathbb{R}^3 : z = 0; y < 1\}.
\]

Observe that for any \(n > 0\), \((0, 1 - 1/n, 0) \in \overline{C}\) but the limit as \(n \to +\infty\), \((0, 1, 0)\), does not belong to \(\overline{C}\). This means that \(\overline{C}\) is not closed, therefore \( C \) is neither closed or open.

(d) \ We want to show that \( D \) is closed. Fix a sequence \((x_n, y_n, z_n) \in D\), which converges to a point \((x_\infty, y_\infty, z_\infty) \in \mathbb{R}^3\): we must prove that \((x_\infty, y_\infty, z_\infty) \in D\). Since \((x_n, y_n, z_n) \in D\),
\[
x_n^2 + 2y_n - z_n = 1;
\]

since \(x^2 + 2y - z\) is the expression of a continuous function, the above equation still holds when passing to the limit \(n \to \infty\): therefore \(x_\infty^2 + 2y_\infty - z_\infty = 1\). This means that \((x_\infty, y_\infty, z_\infty) \in D\). Thus \( D \) is closed.

Problem 2. Which of the following sets are bounded? Compact? Justify.
(a) $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^2 + 3z^2 \leq 4\}$;
(b) $B = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 2y^2 - 3z^2 = 0\}$;
(c) $C = \{(x, y) \in \mathbb{R}^3 : 0 \leq e^x + e^y \leq 2\}$.

(a) If $(x, y, z) \in A$, then $x^2 + 2y^2 + 3z^2 \leq 4$. Thus,

$$x^2 \leq x^2 + 2y^2 + 3z^2 \leq 4, \quad y^2 \leq x^2 + 2y^2 + 3z^2 \leq 4, \quad z^2 \leq x^2 + 2y^2 + 3z^2 \leq 4.$$ 

It follows that $x \in [-2, 2], y \in [-2, 2]$ and $z \in [-2, 2]$. Thus $A$ is bounded. In addition, $A$ is closed: it suffices to apply the same method as Problem 1(b) and (d). Thus, $A$ is compact.

(b) We observe that $B$ cannot be bounded: for any $n$, $(n, n, n)$ is in $B$. This point has norm $n\sqrt{3}$ which goes to infinity as $n$ goes to infinity. Therefore it ends up escaping any given ball. Thus $B$ is not bounded; and not compact.

(c) $C$ cannot be bounded: $(-n, -n)$ belongs to $C$ for all $n \geq 0$. But it escapes any given ball for $n$ large enough.

**Problem 3.** Let $f : A \to \mathbb{R}^2$ be defined by

$$f(x, y) = \frac{1}{x^2 + y^2}, \quad A = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\}.$$ 

Does $f$ have a global maximum? Why doesn’t this contradict the extreme value theorem?

We observe that $f$ does not have a global maximum: indeed, for $x \leq 1/2$, $(x, x) \in A$, and

$$f(x) = \frac{1}{2x^2} \to +\infty \quad \text{as } x \to +\infty.$$ 

This does not contradict the extreme value theorem: indeed, $A$ is not a compact set. To prove this claim, we show that $A$ is not closed. For $n \geq 2$, the sequence $(1/n, 1/n)$ takes values in $A$. However it converges to $(0, 0)$ as $n \to \infty$, and $(0, 0) \notin A$. Therefore $A$ is not closed.

**Problem 4.** Use the extreme value theorem to show that the following functions admit a global minimum and a global maximum:

(a) $f : K \to \mathbb{R}$ defined by

$$K = [2, 4], \quad f(x) = \begin{cases} x^2 & \text{if } 2 \leq x \leq 3 \\ -x + 12 & \text{if } 3 < x \leq 4. \end{cases}$$

(b) $g : L \to \mathbb{R}$ defined by

$L = \{(x, y, z) : x^4 - y^2 - z^2 = 0, \quad y^2 + z^2 \leq 1\}, \quad g(x, y, z) = 4x - e^{zy} + \cos(xy + z).$

(a) $K$ is compact: indeed, it is bounded (it is contained in the ball $B(0, 5)$) and it is closed: if $2 \leq x_n \leq 4$ and $x_n \to x_\infty$, then $x_\infty \in [2, 4]$. Now, $f$ is continuous: although it is defined weirdly, we observe that the only point where there could be a problem is 3. At this point, $3^2 = 9$ and $-3 + 12 = 9$ therefore $f$ can be drawn without raising the pencil. Therefore the extreme value theorem applies and shows that $f$ has a global minimum and a global maximum.
(b) Again, we show that $L$ is compact. It is closed: If $(x_n, y_n, z_n) \in K$ and $(x_n, y_n, z_n) \to (x_\infty, y_\infty, z_\infty) \in \mathbb{R}^3$, then:

$$x_n^4 - y_n^2 - z_n^2 = 0, \quad y_n^2 + z_n^2 \leq 1.$$ 

Passing to the limit does not modify non-strict inequalities or equality (however it modifies strict inequalities in non-strict inequalities, which does not matter here); we deduce that

$$x_\infty^4 - y_\infty^2 - z_\infty^2 = 0, \quad y_\infty^2 + z_\infty^2 \leq 1.$$

Therefore $A$ is closed. To observe that $A$ is bounded, we see that if $(x, y, z) \in K$ then $x^2 + y^2 \leq 1$ and $x^4 = y^2 + z^2$. This implies $x^4 \leq 1$. Thus, $x^2 \leq 1$, and

$$x^2 + y^2 + z^2 \leq 2.$$ 

It follows that $L \subset B(0, 0; 3)$. Hence $L$ is compact. Now, $g$ is continuous because it is classically defined using elementary operations and continuous functions. According to the extreme value theorem, it has both a minimum and a maximum.