HW4 - Due 02/20

Each answer must be mathematically justified. Don’t forget your name.

Problem 1. Are the following sets convex?

(a) \( A = \{ \mathbf{x} \in \mathbb{R}^n : x_1^2 + \ldots + x_n^2 \leq 1 \} \);
(b) \( B = \{ \mathbf{x} \in \mathbb{R}^n : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1, \ldots, 0 \leq x_n \leq 1 \} \);
(c) \( C = \{ (x, y) \in \mathbb{R}^2 : y \geq x^3 - x \} \);
(d) \( D = \{ (x, y, z) \in \mathbb{R}^3 : z = 2x^2 + y^2 \} \).

(a) We first show that the function \( x^2 \) is convex: it is obvious because its second derivative is 2, which is positive. Therefore, for any real \( x, y \) and any \( t \) in \([0, 1]\),

\[ (tx + (1 - t)y)^2 \leq tx^2 + (1 - t)y^2. \]

We apply this to \( x = x_1, \ldots, x_n \) and to \( y = y_1, \ldots, y_n \) to get that for every \( j = 1, \ldots, n \), every \( t \) between 0 and 1,

\[ (tx_j + (1 - t)y_j)^2 \leq tx_j^2 + (1 - t)y_j^2. \]

Sum over \( j \) and get

\[ (tx_1 + (1 - t)y_1)^2 + \ldots + (tx_n + (1 - t)y_n)^2 \leq t(x_1^2 + \ldots + x_n^2) + (1 - t)(y_1^2 + \ldots + y_n^2). \]

Therefore, if \( \mathbf{x} \) and \( \mathbf{y} \) belong to \( A \), that is

\[ x_1^2 + \ldots + x_n^2 \leq 1, \quad y_1^2 + \ldots + y_n^2 \leq 1, \]

then

\[ (tx_1 + (1 - t)y_1)^2 + \ldots + (tx_n + (1 - t)y_n)^2 \leq t(x_1^2 + \ldots + x_n^2) + (1 - t)(y_1^2 + \ldots + y_n^2) \leq t + 1 - t \leq 1; \]

that is, \( t\mathbf{x} + (1 - t)\mathbf{y} \) belongs to \( A \). Therefore \( A \) is convex.

(b) Assume that \( \mathbf{x}, \mathbf{y} \) belong to \( B \). Then,

\[ 0 \leq x_1 \leq 1, \quad 0 \leq y_1 \leq 1, \]
\[ 0 \leq x_n \leq 1, \quad 0 \leq y_n \leq 1. \]

This implies that for any \( t \in [0, 1], \)

\[ 0 \leq tx_1 \leq 1, \quad 0 \leq tx_n \leq 1, \]
\[ 0 \leq (1 - t)y_1 \leq 1, \quad 0 \leq (1 - t)y_n \leq 1. \]

Summing both inequalities gives

\[ 0 \leq tx_1 + (1 - t)y_1 \leq 1, \quad 0 \leq tx_n + (1 - t)y_n \leq 1. \]

Therefore \( t\mathbf{x} + (1 - t)\mathbf{y} \) belongs to \( B \), thus \( B \) is convex.

The best is to draw a picture. This motivates the following choices: \((-1, 0) \in C \) and \((0, 0) \in C \) because \((-1)^3 - (-1) \leq 0 \) and \(0^3 - 0 \leq 0 \). But the middle point \((-1/2, 0)\) is not in \( B \) because \((-1/2)^3 - (-1/2) > 0 \). Thus \( C \) is not convex.

We observe that \((0, 0, 0) \) is in \( D \) and that \((0, 2, 4) \) is in \( D \). However, the middle point \((0, 1, 2) \) is not in \( D \) because \(2 \neq 2 \cdot 0^2 + 1^2 \). Therefore \( D \) is not convex.
Problem 2. Prove the following inequalities:

(a) For any $a, b$ real numbers, $3e^{a+2b} \leq e^{3a} + 2e^{3b}$.
(b) For any $a, b$ real numbers, $(a + b)^{2018} \leq 2^{2017}a^{2018} + 2^{2017}b^{2018}$.

(a) We use that $e^x$ is convex to get $e^{x+2y} \leq \frac{1}{3}e^x + \frac{2}{3}e^y$.

Now take $x = 3a$ and $y = 3b$ and multiply both sides by 3 to get $3e^{a+2b} \leq e^{3a} + 2e^{3b}$.

(b) We use that $x^{2018}$ is convex (its second derivative is positive) to get $\left(\frac{x+y}{2}\right)^{2018} \leq \frac{1}{2}x^{2018} + \frac{1}{2}y^{2018}$.

Then take $x = 2a$ and $y = 2b$ above to get $(a + b)^{2018} \leq 2^{2017}a^{2018} + 2^{2017}b^{2018}$.

Problem 3. Show that if $u(t)$ is convex and increasing and $f(x, y)$ is convex, then the function $g(x, y) = u(f(x, y))$ is convex. Use this fact to prove that the function $h(x, y) = e^{x^2+y^4+1}$ is convex.

If $f$ is convex, then $f(t \vec{x} + (1 - t) \vec{y}) \leq tf(\vec{x}) + (1 - t)f(\vec{y})$.

Apply $u$ on both sides (remark that $u$ is increasing, so the sense of the inequality is preserved) to get $g(t \vec{x} + (1 - t) \vec{y}) \leq u(tf(\vec{x}) + (1 - t)f(\vec{y}))$.

Now, use that $u$ is convex to get $u(tf(\vec{x}) + (1 - t)f(\vec{y})) \leq tu(f(\vec{x})) + (1 - t)u(f(\vec{y}))$.

We deduce that $g(t \vec{x} + (1 - t) \vec{y}) \leq tu(f(\vec{x})) + (1 - t)u(f(\vec{y})) = tg(\vec{x}) + (1 - t)g(\vec{y})$, that is, $g$ is convex. Now in order to show that $h$ is convex, we take $u(t) = e^t$ (which is convex and increasing) and $f(x, y) = 1 + x^2 + y^4$. Note that $f$ is convex: we have $f''(x, y) = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix}$, which is positive semi-definite: $2 > 0$ and $24y^2 \geq 0$. In addition, $h = u \circ f$. Thus by the first part, $h$ is convex.

Problem 4. Find the global minimum of the following functions:

(a) $f(x, y) = e^{x^2+2xy+4y^2}$ over all real $x$ and $y$;
(b) \( g(x, y) = 2x^2 - 2x + 1 - 2xy + y^2 \) over \( \{(x, y) : (x - 1)^2 + (y - 1)^2 \leq 1\} \);
(c) \( h(x, y) = x^2 + y^2 + 1 \) over \( \{(x, y) : 1 \leq x \leq 2, 1 \leq y \leq 2\} \).
(d) (optional) \( i(x, y) = e^{x^2+y^2} + e^{-x^2-y^2} \) over all real \( x \) and \( y \).

(a) We observe that \( f \) is convex: indeed it suffices to prove (by Problem 3) that \( x^2 + 2xy + 4y^2 \) is convex. The Hessian is
\[
\begin{bmatrix}
2 & 2 \\
2 & 8
\end{bmatrix}
\]
which is definite positive. Therefore if \( f \) has a critical point, then it must correspond to the global minimum. The derivatives of \( f \) are
\[
\partial_x f(x, y) = (2x + 2y)e^{x^2+2xy+4y^2}, \quad \partial_y f(x, y) = (2x + 8y)e^{x^2+2xy+4y^2}.
\]
The only critical point is at \((0, 0)\) thus \( f \) has a global minimum at \((0, 0)\), equal to \( f(0, 0) = 1 \).

(b) Again, we first prove that \( g \) is convex, then we look for critical points within the admissible set. The Hessian is
\[
g''(x, y) = \begin{bmatrix}
4 & -2 \\
-2 & 2
\end{bmatrix}
\]
which is positive definite, thus \( g \) is convex. The admissible set is convex (see e.g. Problem 1(a)). The derivatives of \( g \) are
\[
\partial_x g(x, y) = 4x - 2 - 2y, \quad \partial_y g(x, y) = -2x + 2y.
\]
The only critical point is \((1, 1)\). It belongs to the admissible set. Therefore it corresponds to the global minimum of \( g \): \( g(1, 1) = 0 \).

(c) We compute the Hessian:
\[
h''(x, y) = \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}
\]
Therefore \( h \) is convex. Then we look for critical points: there is only one, \((0, 0)\), which is outside the admissible set. Since this admissible set is convex, \( h \) admits its minimum on the boundary of the set. This boundary is given by four pieces of line, corresponding to \((1, 1 + t); (1 + t, 1); (1 + t, 2) \) and \((2, 1 + t)\), where \( t \) lies between 0 and 1. Therefore we must look for the minimum of the following functions:
\[
1 + (1 + t)^2 + 1; \quad (1 + t)^2 + 1 + 1; \quad (1 + t)^2 + 2^2 + 1; \quad 2^2 + (1 + t)^2 + 1
\]
for \( t \) between 0 and 1. Clearly, the last ones are bigger than the first one; and the first equal \( 3 + 2t + t^2 \). This function decreases for \( t \geq -1 \) and increases after. Therefore \( 3 + 2t + t^2 \) is minimal for \( t \) in \([0, 1]\) when \( t = 0 \): 3 is its global minimum. It follows that 3 is the global minimum of \( h \) (attained at \((1, 1)\)).

(d) We check that \( i \) is convex:
\[
\partial_x i(x, y) = 2x(e^{x^2+y^2} - e^{-x^2-y^2}), \quad \partial_y i(x, y) = 2y(e^{x^2+y^2} - e^{-x^2-y^2}),
\]
\[
i''(x, y) = \begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}(e^{x^2+y^2} - e^{-x^2-y^2}) + \begin{bmatrix}
4x^2 & 2x \\
2y & 2y
\end{bmatrix}(e^{x^2+y^2} + e^{-x^2-y^2}).
\]
We observe that the second matrix is positive semi-definite while the first one is positive definite (because $e^t \geq e^{-t}$). Therefore $i$ is convex. We observe that $(0,0)$ is a critical point of $i$: its derivatives vanish simultaneously at $(0,0)$. Hence it must correspond to the global minimum of $i$: $i(0,0) = 2$.

**Problem 5.** For which values of $a$ and $b$ is the function $f(x, y) = e^{ax^2 + by^2}$ convex?

We compute the Hessian of $f$:

$$
\partial_x f(x, y) = 2axe^{ax^2+by^2}, \quad \partial_y f(x, y) = 2bye^{ax^2+by^2},
$$

$$
f''(x, y) = \begin{bmatrix}
2a + (2ax)^2 & 4abxy \\
4abxy & 2b + (2by)^2
\end{bmatrix} e^{ax^2+by^2}.
$$

Now we test when this is positive semidefinite: we must have $2a + 4ax^2 \geq 0$ for all $x$. This is achieved iff $a \geq 0$ (take $x = 0$). We must also have for all $x$ and $y$,

$$(2a + 4a^2x^2)(2b + 4b^2y^2) - 4abxy \cdot 4abxy = 4ab + 8a^2bx^2 + 8b^2ay^2 \geq 0,$$

which can happen only if $b \geq 0$ (take $x = y = 0$). When $a$ and $b$ are both nonnegative, $4ab + 8a^2bx^2 + 8b^2ay^2 \geq 0$. Thus $f$ is convex.

**Problem 6.** Show that if $f$ is a convex smooth function on $\mathbb{R}$, then for any value of $a$, the graph of $f$ is above the tangent line at $a$. (Hint: use the fundamental theorem of calculus).

Fix $a$ a point. By the fundamental theorem of calculus:

$$f(x) = f(a) + \int_a^x f'(t)dt.$$

Now use that $f'$ is non-decreasing (because $f$ is convex): if $x \geq a$ and $t$ is between $a$ and $x$, $f'(t) \geq f'(a)$. It follows that

$$f(x) = f(a) + \int_a^x f'(t)dt \geq f(x) = f(a) + \int_a^x f'(a)dt \geq f(a) + (x-a)f'(a).$$

This means that $f$ is above its tangent line for all $x \geq a$. For $x \leq a$ and $t$ between $x$ and $a$, $f'(x) \leq f'(a)$ and since the bound in the integral are inverted,

$$f(x) = f(a) + \int_a^x f'(t)dt \geq f(x) = f(a) + \int_a^x f'(a)dt = f(a) - \int_x^a f'(t)dt \geq f(a) - \int_a^x f'(a)dt = f(a) - (a-x)f'(a) = f(a) + (x-a)f'(a).$$

This shows that $f$ is above its tangent line for $x$ before $a$. 