Problem 1. Which of the following sets are linearly independent?

(a) \[
\begin{bmatrix}
4 & -1 & 7 \\
3 & 0 & 3 \\
1 & 2 & 4
\end{bmatrix},
\begin{bmatrix}
-9 & -9 \\
-6 & -9 \\
1 & 2 & 4
\end{bmatrix}
\]

(b) \[
\begin{bmatrix}
\pi & \sqrt{5} \\
0 & \pi^2 \\
\cos(5) & -\sqrt{3}
\end{bmatrix},
\begin{bmatrix}
\pi & 2 \\
\ln(3) & \sqrt{2} \\
\cos(2) & -1
\end{bmatrix}
\]

(c) \[
\begin{bmatrix}
3 & -4 & 1 \\
-2 & 5 & 3 \\
1 & 1 & 2
\end{bmatrix}
\]

(a) Form the matrix of the three vectors and perform elementary operations:

\[
\begin{bmatrix}
4 & -1 & 7 \\
3 & 0 & 3 \\
1 & 2 & 4
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & -9 & -9 \\
0 & 2 & 3 \\
0 & 1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
1 & 2 & 4
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 2 & 4 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}.
\]

This has triangular form with non-zero elements on the diagonal, therefore the three vectors are linearly independent.

(b) These are four 3-vectors. Therefore their span is of dimension at most 3. Thus these vectors cannot span a four-dimensional space: they are not linearly independent.

(c) Again, form the matrix of the three vectors and perform elementary operations:

\[
\begin{bmatrix}
3 & -4 & -1 \\
-2 & 5 & 3 \\
1 & 1 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & -7 & -7 \\
0 & 7 & 7 \\
0 & 1 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
1 & 1 & 2
\end{bmatrix}.
\]

One of the lines is made of zeroes, therefore these vectors are not linearly independent.

Problem 2. Find the global minima/maxima of the following functions (if they exist):

(a) \[f(x) = \cos(2x) + 5, \ x \in \mathbb{R};\]

(b) \[g(x) = x + \frac{1}{x + 2}, \ x \neq -2;\]

(c) \[h(x) = x^2 + 4x - 3, \ x \in [0, 4];\]

(d) \[i(x) = x^4 - 4x^2, \ x \in \mathbb{R};\]

(e) \[j(x) = |x + 1| + (x + 1)^2 - 3, \ x \in \mathbb{R}.\]

(a) We recall that \(-1 \leq \cos(y) \leq 1\). Thus \(4 \leq \cos(2x) + 5 \leq 6\). In addition, \(f(0) = 6\) and \(f(\pi) = 4\). Therefore the global minimum of \(f\) is 4 and the global maximum is 6.

(b) We observe that

\[
\lim_{x \to -2^-} g(x) = -\infty, \quad \lim_{x \to -2^+} g(x) = +\infty,
\]

thus \( g \) has neither a global maximum or minimum.

(c) We compute \( h'(x) : h'(x) = 2x + 4 \). It vanishes at \( x = -2 \) and it is positive for \( x > -2 \). Therefore, \( h \) is increasing on \([0, 4]\). It follows that the global minimum of \( h \) is \( h(0) = -3 \) and the global maximum is \( h(4) = 29 \).

(d) Since the limit of \( i \) at \(+\infty\) and \(-\infty\) is \(+\infty\), thus \( i \) cannot have a global maximum.
We look for a global minimum.
We compute \( i'(x) : i'(x) = 4x^3 - 8x = 4x(x - \sqrt{2})(x + \sqrt{2}) \). It vanishes at \( 0, \sqrt{2}, -\sqrt{2} \). A sign study shows that \( i \) decreases on \((-\infty, -\sqrt{2}]\), increases on \([-\sqrt{2}, 0]\), decreases on \([0, \sqrt{2}]\) and increases after \([\sqrt{2}, \infty)\). Thus we have local minima at \( \sqrt{2}\) and \( -\sqrt{2}\) (and a local maximum at \( 0 \)). Since \( i(\sqrt{2}) = i(-\sqrt{2}) = -4 \), \(-4\) is the global minimum.

(e) Again, the limit of \( j \) at \(+\infty\) and \(-\infty\) is \(+\infty\), thus \( i \) cannot have a global maximum.
We observe that \(|y| \geq 0\) and \( y^2 \geq 0 \). Thus, \( j(x) \geq -3 \). In addition, \( j(-1) = -3 \) thus the global minimum of \( j \) is \(-3\).

**Problem 3.** Solve the linear system
\[
\begin{cases}
-2x + y + z = 0 \\
-x + z = 0 \\
x + 2y - 3z = 0
\end{cases}
\]

We need to find the null space of the matrix
\[
\begin{bmatrix}
-2 & 1 & 1 \\
-1 & 0 & 1 \\
1 & 2 & -3
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 5 & -5 \\
0 & 2 & -2 \\
1 & 2 & -3
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & -1 \\
1 & 2 & -3
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & -1 \\
1 & 0 & -1
\end{bmatrix}
\]
Therefore this system reduces to \( y - z = 0 \) and \( x - z = 0 \). The solutions are triplets \((x, x, x)\).

**Problem 4.** Find the rank of the following matrices:

(a) \[
\begin{bmatrix}
4 & -1 & 2 \\
2 & 3 & 1
\end{bmatrix} ; \quad \text{(b)} \quad \begin{bmatrix}
-1 & 0 & -4 \\
2 & 3 & 11
\end{bmatrix} ; \quad \text{(c)} \quad \begin{bmatrix}
1 & 2 & 4 \\
1 & 2 & 4
\end{bmatrix}
\]

(a) Perform elementary operations:
\[
\begin{bmatrix}
4 & -1 & 2 \\
2 & 3 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
4 & -1 & 4 \\
2 & 3 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & -1 & 4 \\
0 & 3 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & -1 & 0 \\
0 & 3 & 14
\end{bmatrix}
\]
This matrix has rank 2.

(b) Perform elementary operations:
\[
\begin{bmatrix}
-1 & 0 & -4 \\
2 & 3 & 11
\end{bmatrix} \rightarrow \begin{bmatrix}
-1 & 0 & 0 \\
2 & 3 & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
-1 & 0 & 0 \\
2 & 3 & 0
\end{bmatrix}
\]
This matrix has rank 2.

(c) This matrix has three identical non-zero lines, thus it has rank one.
Problem 5. Find all values of $a$ such that the system
\[
\begin{align*}
x + 2y &= 0 \\
y + 2z &= 0 \\
z + 2x &= a
\end{align*}
\]
has exactly one solution.

In matrix form, this system looks like
\[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & 2 \\
2 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
a
\end{bmatrix}.
\]

Let us find the rank of the matrix on the LHS:
\[
\begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & 2 \\
2 & 0 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & 2 \\
0 & -4 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & 2 \\
0 & 0 & 9
\end{bmatrix}.
\]

This matrix has rank 3. Therefore it is invertible. It follows that for any value of $a$, the above system has a unique solution – which is given by
\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
1 & 2 & 0 \\
0 & 1 & 2 \\
2 & 0 & 1
\end{bmatrix}^{-1}
\begin{bmatrix}
0 \\
0 \\
a
\end{bmatrix}.
\]

Problem 6. Find every value of $a$ such that the matrix
\[
\begin{bmatrix}
-1 & 3 & 2 \\
2 & a & 2 \\
-5 & 5 & a
\end{bmatrix}
\]
has maximal rank.

To find the rank of this matrix, we perform elementary operations:
\[
\begin{bmatrix}
-1 & 3 & 2 \\
2 & a & 2 \\
-5 & 5 & a
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-1 & 3 & 2 \\
0 & a+6 & 6 \\
0 & -10 & a-10
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-1 & 1 & 2 \\
0 & a & 6 \\
0 & -a & a-10
\end{bmatrix}
\rightarrow
\begin{bmatrix}
-1 & 1 & 2 \\
0 & a & 6 \\
0 & 0 & a-4
\end{bmatrix}.
\]

The above matrix has maximal rank when all diagonal elements are non-zero, which is satisfied for $a \neq 0, 4$.

Problem 7. Among all rectangles of perimeter 20, what are the dimension of the one with maximal area?

Let $\ell$ and $L$ be the width and length of a rectangle of perimeter 20. Then $2(\ell + L) = 20$, which implies $L = 10 - \ell$. In addition, the area of this rectangle is
\[\ell \cdot L = \ell(10 - \ell)\]

The problem asks for this area to be maximal, therefore we maximize the function $f(\ell) = \ell(10 - \ell)$. We observe that $f'(\ell) = 10 - 2\ell$, therefore $f$ is increasing for $\ell \geq 5$.
and decreasing for $\ell \geq 5$. It has a global maximum at $\ell = 5$. This proves that the rectangle of perimeter 20, with maximal area have dimensions $5 \times 5$. 