Problem 1. Find the general solutions of the following ODEs:

(a) \( y' + 2te^{t^2} + y = 0 \);
(b) \( y' + (3 - y)(y - 2) = 0 \);
(c) \( y' = y - 2y^2 \);
(d) \((2y + 3)y' + y^2 + 3y + 2 = 0\).

(a) We can divide both sides by \( e^{-y} \) (this is a positive number) to get:
\[ y'e^{-y} + 2te^{t^2} = 0. \]
Split \( dy \) and \( dt \) to obtain
\[ e^{-y}dy + 2te^{t^2}dt = 0. \]
This integrates to
\[ e^{-y} + e^{t^2} + C = 0. \]
Thus (when defined):
\[ y = -\ln(e^{t^2} + C), \text{ where } C \text{ is any constant}. \]

(b) We first observe that if \( y \) reaches the value 2 or 3 for some \( t \), then \( y = 2 \) or \( y = 3 \) for every \( t \). Therefore, we can put aside these two solutions for now; this allows us to divide by \((3 - y)(2 - y)\). We then split \( dy \) and \( dt \) and get
\[ \frac{dy}{(3 - y)(y - 2)} + dt = 0 \Rightarrow \int \left( \frac{1}{3 - y} + \frac{1}{y + 2} \right) dy + \int dt \Rightarrow -\ln |3 - y| + \ln |y - 2| + t + C = 0. \]
We deduce that
\[ \ln \left| \frac{y - 2}{3 - y} \right| + t + c = 0 \Rightarrow \left| \frac{y - 2}{3 - y} \right| e^{t+c} = 1 \Rightarrow y - 2 = 3 - y \text{ or } y - 2 = -(3 - y). \]
Therefore, writing \( A = e^c \) or \( A = -e^c \), observe that \( A \) can then be any non-zero number – then
\[ \frac{y - 2}{3 - y} = Ae^{-t} \Rightarrow y = \frac{2 - 3Ae^{-t}}{1 - Ae^{-t}}, \text{ where } A \text{ is any number}. \]
We deduce that the general solution is
\[ y = 2 \text{ or } y = 3 \text{ or } y = \frac{2 - 3Ae^{-t}}{1 - Ae^{-t}}, \text{ where } A \text{ is any number}. \]

(c) We adopt the same strategy as in (b). We first observe that \( y = 0 \) and \( y = 1/2 \) are solutions. Then we are allowed to divide by \( y - 2y^2 \). Splitting \( dy \) and \( dt \) yields
\[ \frac{dy}{y(1 - 2y)} = dt \Rightarrow \int \left( \frac{1}{y} + \frac{2}{1 - 2y} \right) dy = 0 \Rightarrow \ln |y| - \ln |1 - 2y| = t + C. \]
We then exponentiate everything and remove the absolute value, as above, to get
\[
\frac{y}{1-2y} = Ae^t \Rightarrow y = \frac{Ae^t}{1+2Ae^t} \quad \text{where } A \text{ is any number.}
\]
We conclude that the general solution is
\[
y = 0 \text{ or } y = 1/2 \text{ or } y = \frac{Ae^t}{1+2Ae^t} \quad \text{where } A \text{ is any number.}
\]

(d) As above, we observe that \( y = -1, -2 \) are solutions. We can then put them aside and divide by \( y^2 + 3y + 2 \). This yields
\[
\frac{(2y+3)dy}{y^2 + 3y + 2} = dt \Rightarrow \int \left( \frac{1}{y+1} + \frac{1}{y+2} \right) dy + \int dt = 0 \Rightarrow \ln |(y+1)(y+2)| + t + C = 0.
\]
We then exponentiate:
\[
|(y+1)(y+2)|e^{t+c} = 1.
\]
We get rid of the absolute value:
\[
(y+1)(y+2) = Ae^{-t}, \quad \text{where } A \text{ can be any number.}
\]
We solve for \( y \) – that’s a bit technical – and end up with the following general solution:
\[
y = \frac{-3 \pm \sqrt{1+4Ae^{-t}}}{2} \text{ or } y = 1 \text{ or } y = 2.
\]

Problem 2. Solve the system of ODEs
\[
\begin{align*}
x' &= 5x - 6y - 6z \\
y' &= 3x - 4y - 3z \\
z' &= -z
\end{align*}
\]
We write the system in matrix form: \( A = PDP^{-1} \) where
\[
A = \begin{bmatrix} 5 & -6 & -6 \\ 3 & -4 & -3 \\ 0 & 0 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 2 \\ -1 & 2 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}
\]
Therefore as in class, we deduce that the general solution is
\[
Ae^{2t} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + e^{-t} \begin{bmatrix} B \\ 2B - C \\ -B + C \end{bmatrix}.
\]

Problem 3. (a) Find the general solution to the problem
\[
y'' + 4y' + 13y = 0.
\]
(b) Find a solution to the problem
\[
y'' + 4y' + 13y = 18e^t.
\]
(c) Find the solution to the problem
\[ y'' + 4y' + 13y = 18e^t, \quad y(0) = 2, \quad y'(0) = 6. \]

(a) We compute the discriminant of the characteristic equation: it is \(16 - 52 = -36 < 0\). Therefore we have two solutions \(e^{-2t}\cos(3t)\) and \(e^{-2t}\sin(3t)\). It follows that the general solution is
\[ e^{-2t}(A\cos(3t) + B\sin(3t)) \]
where \(A, B\) are any constants.

(b) We look for a solution of the form \(Ce^t\). We get \(18C = 18\) therefore \(C = 1\). It follows that \(e^t\) is solution.

(c) The general solution is obtained by summing (a) and (b):
\[ e^{-2t}(A\cos(3t) + B\sin(3t)) + e^t \]
where \(A, B\) are any constants. Two find \(A\) and \(B\) it suffices to formulate the corresponding equations:
\[
\begin{align*}
2 &= A + 1 \\
6 &= -2A + 3B + 1
\end{align*}
\]
We find \(A = 1, B = 7/3\) therefore the solution is \(e^{2t}(\cos(3t) + 7\sin(3t)/3) + e^t\).

**Problem 4.** Find all the solutions to the ODE
\[ y'' - e^{2y} = 0, \quad y(0) = 0, \quad y'(0) = 1. \]
(Hint: multiply the equation by \(y'\) first).

We first multiply by \(2y'\) and get
\[ 2y'y'' - 2y'e^{2y} = 0. \]
By the chain rule, this can be written as \([y'^2 - (e^{2y})'] = 0\). We deduce that
\[ y'^2 - e^{2y} = C \]
Since \(y'(0) = 1\) and \(y(0) = 0\), this simplifies to \(y'^2 - e^{2y} = 0\). Therefore \(y' + e^y = 0\) or \(y' - e^y = 0\). In order to solve these two ODEs, we divide by \(e^y\) and split \(dy\) and \(dt\), obtaining
\[ e^{-y}dy + dt = 0 \text{ or } e^{-y}dy - dt = 0 \Rightarrow -e^{-y} + t + C = 0 \text{ or } -e^{-y} - t + C = 0. \]
It follows that
\[ y = -\ln(t + C) \text{ or } y = -\ln(-t + C). \]
Now, \(y(0) = 0\) implies \(C = 1\) in both the first case and the second case. But the derivative of \(-\ln(t + 1)\) at \(t = 0\) is \(-1\) while the derivative of \(-\ln(-t + 1)\) at \(t = 0\) is \(1\). We deduce that the solution of this ODE is \(y = -\ln(-t + 1)\).

**Problem 5.** Solve the system of ODEs
\[
\begin{align*}
y'' &= y + 3z \\
z' &= 2z
\end{align*}
\]
We first solve for $z$. We find $z = Ae^{2t}$. Now the equation in $y$ becomes

$$y'' = y + 3Ae^{2t}.$$ 

We observe that $y = Be^t + Ce^{-t}$ is the general solution to the homogeneous equation $y'' = y$. We now look for a particular solution. We look for it in the form $y = De^{2t}$. A quick calculation yields $4D = D + 3A$ therefore $A = D$ and $y = Be^t + Ce^{-t} + Ae^{2t}$ is the general solution for $y$. We deduce that the general solution for the system is

$$\begin{cases} y = Be^t + Ce^{-t} + Ae^{2t} \\ z = Ae^{2t} \end{cases}$$

where $A, B, C$ can be any constants.