THE F_4 AND E_6 FAMILIES HAVE A FINITE NUMBER OF POINTS

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Abstract. Deligne conjectured that there is a continuous family of exceptional Lie algebras, a category with a continuous parameter that specializes to the category of representations of the exceptional Lie algebras when the parameter is specialized to a particular value. There are related sub-exceptional series for each of the other rows of Freudenthal's magic square, ending in F_4, E_6, and E_7. We show that there is no F_4 or E_6 series of Lie algebras depending on a continuous parameter by showing that the fundamental relation for each series implies that a polynomial in the parameter is equal to 0. The roots of these polynomials correspond to known evaluations.

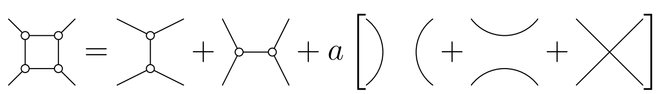
1. Introduction

Deligne [5, 6] conjectured that there is a 1-parameter family containing all the exceptional Lie algebras, by analogy with the 1-parameter families of classical Lie algebras. For instance, the $\mathfrak{sl}_n$ Lie algebras have uniform decomposition formulas as n varies, parametrized by partitions. (For any given value of n, many of the representations will vanish.) Vogel [11] later extended this conjecture to various conjectures about a 2-parameter family involving all Lie algebras.

The conjectures are based on two pieces of evidence:

- An identity between tensors that hold uniformly across the family, depending only on a single parameter λ. This was noticed by Vogel [10] and was the original motivation for Deligne's work.
- Uniform decompositions of tensor products of the adjoint representations into irreducible modules for a suitable form of the Lie group; the dimension of the modules is a rational function of λ, and usually factors as a product of linear factors.

Deligne's conjecture was that there is a semi-simple symmetric monoidal abelian category with duals over $\mathbb{Q}(t)$ with certain restrictions, in particular that the category is semi-simple and that there is an "adjoint" object playing the role of the adjoint representation in a Lie algebra. In particular, in any such category we can assign a value to any vertex-oriented trivalent graph. Although Deligne did not explicitly conjecture it, presumably one should add to the conjecture that the evaluation in this category should also satisfy Vogel's fundamental identity, which can be written:



One natural way to arrive at this relation is to say that the fourth symmetric power of the adjoint representation has a 1-dimensional invariant subspace. Symmetrizing the left side of this equation, and you get one element of this subspace; symmetrizing the right hand side, the first two terms disappear, and you get another element; there must be some linear relation

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between the two. This condition characterizes the exceptional series among all simple Lie algebras.

This whole story caries over to the *sub-exceptional series*, corresponding to the other lines in Freudenthal's magic square, with different conjectural categories for each row. We show that the tensor identity for the F_4 and E_6 families implies that the parameter must take one of a finite number of values. In particular, this disproves the stronger analogue of Deligne's conjecture for these series. Since these families appear to be closely analogous to the exceptional family, this provides evidence against Deligne's conjecture.

One positive bit of evidence is that in the degrees examined so far, it is possible to evaluate every graph using the fundamental relations of each series.

Note that from the geometric point of view, Landsberg and Manivel found [9] that F_4 row and, to a lesser extent, the E_6 row of the magic square did not behave as well as the E_7 and exceptional (E_8) rows.

The graphical techniques that we use to treat the various exceptional series were pioneered by Cvitanović [2, 3, 4, 1], who first investigated these series in the 1970's.

Acknowledgements. Many thanks to Predrag Cvitanović, J.M. Landsberg, Pierre Deligne, and Bruce Westbury for explaining to me much of the background of this paper.

2. The F_4 series

As Deligne and Gross [7] explained, the F_4 series is best thought of as a series of Lie algebras h with a representation W of dimension n that admits a Lie bracket (in the exceptional series) when summed with the exceptional Lie algebra g_2 and some extra factors: the vector space

$$g_2 \oplus h \oplus (V \otimes W),$$

where V is the defining 7-dimensional representation of g_2, should have a Lie bracket, containing g_2 and h as subalgebras acting in the natural ways, and an invariant nondegenerate symmetric bilinear form; since V has a symmetric bilinear form, W must as well.

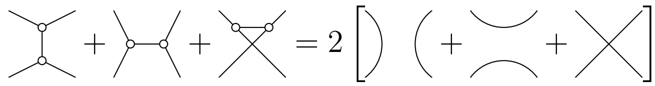
The possible components of the bracket are easy to write down and include the bracket on g_2 and on h, the representation of g_2 on V and V^*, and the representation of h on W and W^*. The most interesting component of the bracket is a map

$$\wedge^2(V \otimes W) \to V \otimes W$$

In $V \otimes V \to V$, there is a polarized version of the invariant antisymmetric cubic tensor. To pair with it, we must have a symmetric cubic tensor in W.

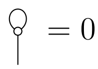
Thus all Lie groups in the F_4 series have an orthogonal representation with an invariant symmetric cubic tensor. (This is the start of the uniform decomposition of this basic representation.) We can use this to associate a rational number to any trivalent graph G: tensor together a copy of the invariant tensor in $\mathrm{Sym}^3(W)$ for each vertex in G, and use the dual of the invariant form in $\mathrm{Sym}^2(W^*)$ to contract tensor indices for the vertices at the end of each edge of G. We can do this for graphs with k univalent ends as well, leaving not an element of the base field but an element of $W^{\otimes k}$.

This Lie bracket that we constructed also has to satisfy the Jacobi relation. This is automatic for almost all components of the relation; the exception is the component mapping $(V \otimes W)^{\otimes 3}$ to $V \otimes W$. An easy computation shows that the Jacobi relation is true iff the following identity between tensors in $W^{\otimes 4}$ is satisfied:

(F4fund) 

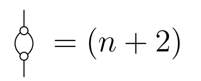
The factor of 2 is there only to make some of the denominators smaller; it can be changed by rescaling the vertex. This is the fundamental relation for the F_4 family.

It is also convenient to assume that a self-loop on a vertex is equal to 0:

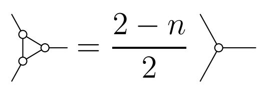
(F4loop1) 

Without this assumption, there is only one additional solution, with $n = 1$ and a barbell equal to 2.

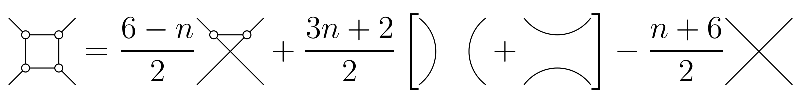
Pairing two of the legs in (F4fund) with the inner product, we get a reduction for a cycle of length 2:

(F4loop2) 

Likewise, attaching the symmetric cubic to two legs of (F4fund), we get a reduction for a cycle of length 3:

(F4loop3) 

A little more work takes care of a cycle of length 4 as well:

(F4loop4) 

Thus these relations allow us to reduce any diagram with a cycle of length less than 5 (girth less than 5) to a sum of simpler diagrams with fewer vertices. This allows us to evaluate a large number of diagrams; the smallest diagram that cannot be immediately evaluted is the Peterson graph with 10 vertices, which has girth 5. However, the fundamental relation also gives us relations among the values of these graphs, and in particular lets us evaluate the Peterson graph. By a computer program which searches for relations between trivalent graphs of girth ≥ 5, reducing any graph with smaller girth, we can try to see if

- any graph can be evaluated by this relation, and
- whether the evaluation is consistent for all values of n.

See Table 1 for the number of graphs we need to consider.

Up to 16 vertices, the answer to the first question is affirmative, but the answer to the second question is negative: linear relations among graphs with 16 vertices imply that the following polynomial of degree 9 in n is zero:

$$(n-26)(n-14)(n-8)(n-5)(n-2)^2 n(n+1)(n+2) = 0$$

The factors have the following interpretations as representations of particular groups or supergroups:

No. vertices	No. graphs: trivalent girth ≥ 5	No. graphs: trivalent, bipartite girth ≥ 6
≤ 8	0	0
10	1	0
12	2	0
14	9	1
16	49	1
18	455	3
20	5783	10
22	90938	28
24	1620479	162
26	31478584	1201
28	656783890	11415
30	?	125571

Table 1. The size of the spaces we have to consider for linear relations. Note that the number of graphs starts small and grows rapidly. The second column is relevant for the F4 series, the third column is relevant for the E6 series. Data courtesy of Gordon Royle.

Group G	n	$\dim \mathfrak{g}$	η	Notes
F_4	26	52	6	Defining representation
$C_3 = Sp(6)$	14	21	4	$\wedge^2$ of defining $\ominus$ symplectic form
$A_2 = SU(3)$	8	8	3	
$A_1 = SU(2)$	5	3	5/2	$\wedge^3$ of defining
Finite S_3	2	0	2	Occurs twice
trivial	0	0	5/3	All closed graphs give 0
$OSp(1 \mid 2)$	-1	1	3/2	defining
$SU(2)$	-2	3	4/3	Vertex is 0; both sides of relation vanish

Here η is closely related to the ν parameter of Deligne and Gross: $\eta = \nu + 1$. The reason for using η rather than ν is that the dimension of the defining representations is a linear function of η, while the automorphism of the exceptional family interchanging the two non-trivial eigenvalues of the Casimir on the symmetric square of the adjoint is $\eta \mapsto 1/\eta$.

The significance of the doubled factor is not yet clear; presumably there is some invariant valued in $\mathbb{Q}[\epsilon]/(\epsilon^2)$ at $n = 2$.

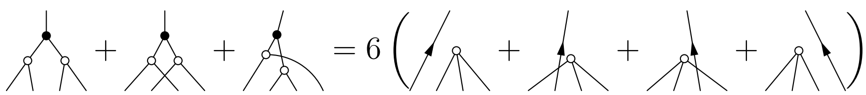
Note that a connected trivalent graph with $2k$ vertices can give a polynomial of degree at most $k + 1$ in n; thus 16 vertices is the first time we could hope to get this polynomial relation. In fact, the program quickly finds the relation when looking at graphs with 16 vertices; not all 49 graphs need to be considered.

3. The E_6 series

The E_6 series of Lie algebras and representations (h, W) is similar to the F_4 series, except that we add a_2 in order to find a group in the exceptional series: the vector space

$$a_2 \oplus h \oplus (V \otimes W) \oplus (V^* \otimes W^*)$$

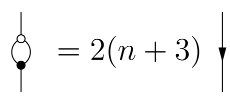
should have a Lie bracket, where V is the defining 3-dimensional representation of a_2. Note that V is not self-dual, so W need not be self-dual as well. Since there is a canonical element of $\wedge^3 V$ and $\wedge^3 V^*$, we should have a canonical element of $S^3 W$ and $S^3 W^*$ to get the antisymmetric Lie bracket. This lets us assign a value to any bipartite trivalent graph, with vertices colored white and black: tensor together a copy of the tensor in $S^3 W$ for each white vertex and the tensor in $S^3 W^*$ for each black vertex, and pair the indices corresponding to the edges of the graph. This time the Jacobi relation on the larger algebra requires that the following relation (and the dual with colors switched) hold:



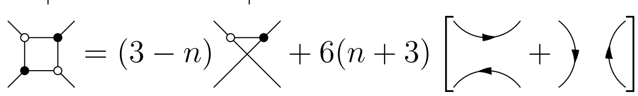
Note that each side is the symmetrization of a single graph under an S_4 action; one natural way to arrive at this condition is to require that there is a 1-dimensional space of maps from $S^4(V)$ to V.

With some mild non-degeneracy condition (implied by simplicity of the representation W), this implies the following reductions for loops of length 2 and 4:

(1)



(2)



Since the graphs are bipartite, the shortest allowable remaining cycle is length 6. See Table 1 for the number of relevant graphs.

Graphs with up to 22 vertices can be evaluated based on these relations, but relations among graphs with 22 vertices imply that the following polynomial of degree 12 in n is 0:

$$(n-27)(n-15)(n-9)(n-6)(n-3)^2(n-1)n^2(n+1)(n+3)^2 = 0$$

Again, the factors have natural interpretations as members of the E_6 family:

Group G	n	$\dim \mathfrak{g}$	η	Notes
E_6	27	78	6	
A_5 or $SL(6)/\pm 1$	15	35	4	$\wedge^2$ of defining
$2A_2$	9	16	3	
A_2	6	8	5/2	
$2\mathbb{C}$	3	2	2	Occurs twice
0	1	0	5/3	
0	0	0	3/2	Occurs twice; all invariants are 0
$OSp(1 \mid 2)$	-1	1	4/3	
A_2	-3	8	1	Occurs twice; the vertex is 0

Again, notice that graphs with 22 vertices are the smallest that could give a polynomial relation of this degree. When comparing this with the F_4 table, you should compare lines with the same η value.

4. Going further

Each row in the magic triangle gives a graphical calculus satisfying a uniform set of equations like the two above. For the rows above F_4 (ending with D_4, G_2, A_2, and A_1), it is a straightforward calculation that can be done by hand to show that there is a polynomial in n with a small number of roots that must be 0 for consistency. For the E_7 and E_8 rows, it would be interesting to see if known relations are consistent and complete, but a computer search as we used is not likely to be helpful, due to the degree of the polynomial obtained.

For the exceptional (E_8) series in Deligne's original conjecture, for instance, there are at least 8 known members of the family, each of which gives two roots of the polynomial due to the symmetry, so the polynomial must be of degree at least 16 (and probably more), which would require a trivalent graph with at least 30 vertices. The relation mentioned at the beginning lets us assume that there are no 4 cycles, and another relation lets us reduce 5 cycles, so we need consider graphs with girth at least 6; but there are 122,090,544 trivalent graphs with 30 vertices and girth at least 6, well beyond the range of any linear solver.

One interesting feature of the above tables is that Lie algebras related to the sextonions [8, 12] are missing. They do not satisfy some of the assumptions: for instance, the fundamental representation is not simple. (It is indecomposable but not irreducible.) This might be a clue as to how to modify the conjecture to produce an honest family of Lie algebras, or it might be that the suitable modification allows only the one additional point in each of these families.

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