

Characterizing generic global rigidity

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<http://www.math.columbia.edu/~dpt/speaking>

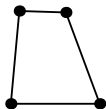
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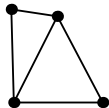
Flavors of rigidity

- ▶ A *framework* in \mathbb{E}^d is a graph and a map from its vertices to \mathbb{E}^d .
- ▶ A framework is *locally rigid in \mathbb{E}^d* if every other framework in a small neighborhood with same edge lengths is related to it by an isometry of \mathbb{E}^d .
- ▶ A framework is *globally rigid in \mathbb{E}^d* if every other framework in \mathbb{E}^d with same edge lengths is related to it by an isometry of \mathbb{E}^d .

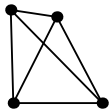
2D Examples



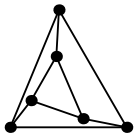
Not locally rigid



Locally but not globally rigid



Globally rigid



Locally but not globally rigid

Aside: Simplices

Theorem (Asimow-Roth '78)

Any framework whose graph is a complete graph is globally rigid.

A framework with $d + 1$ or fewer vertices in d dimensions is locally rigid iff graph is complete.

Thus no interesting questions for graphs with few vertices.

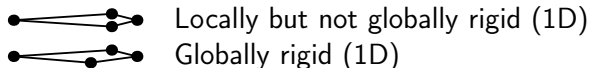
Will assume graphs have at least $d + 2$ vertices.

Rigidity is NP-hard...

Theorem (Saxe '79)

Checking whether a framework with integer coordinates is globally rigid is NP-hard.

Idea: In 1D, need to solve a partition problem.



So problem seems hopeless!

... Generic rigidity is easy

Definition

A framework is *generic* if its coordinates do not satisfy any polynomial equation.



Definition

A graph is *generically globally rigid* if a generic framework of it is globally rigid.

We

- ▶ characterize generically globally rigid graphs;
- ▶ show global rigidity is independent of the generic framework; and
- ▶ give an efficient randomized algorithm for checking the condition.

History and applications

2D case understood completely (Laman '70, Lovász-Yemini '82, Jackson-Jordán '05).

Hendrickson '92 gave simple necessary conditions for global rigidity;
Connelly showed these were not sufficient in 3D.

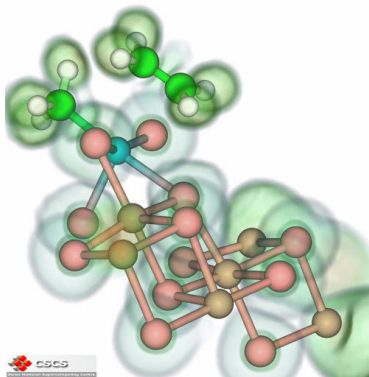
People care!

Reconstruction: given some distances
between nodes, reconstruct
framework.

Global rigidity necessary to be
well-posed.

Applications:

- ▶ molecular chemistry
- ▶ sensor networks



Geometry of maps

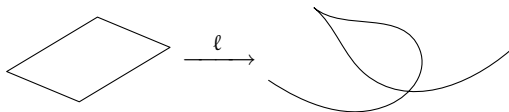
Definition

The *length-squared function* ℓ is the map from frameworks of a graph to its edge lengths, squared.

Definition

The *rigidity matrix* of a framework is the Jacobian ℓ^* of ℓ .

The *rank* of an algebraic map is the rank of its linearization at generic points (= dimension of the image).



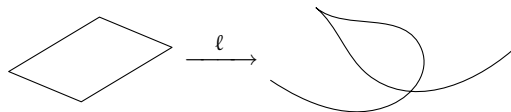
Generic local rigidity

Theorem (Asimow-Roth '78)

A graph is generically locally rigid \Leftrightarrow rank of ℓ^* is generically $vd - \frac{d(d+1)}{2}$.

Interpretation:

- ▶ Group $\text{Eucl}(d)$ of isometries has dimension $\frac{d(d+1)}{2}$.
- ▶ Kernel of ℓ^* always contains tangent to $\text{Eucl}(d)$.
- ▶ Graph is generically locally rigid iff $\ker \ell^*$ no bigger.

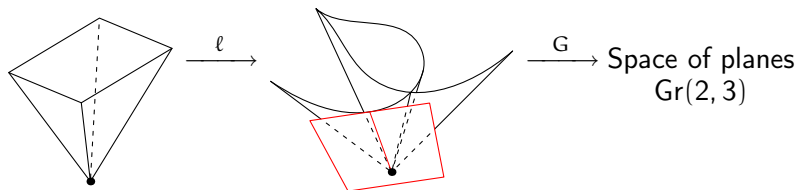


The Gauss map

Definition

The *measurement set* M is the image of ℓ .

The *Gauss map* G of a *homogeneous* semi-algebraic set of dimension t in \mathbb{R}^n takes each *smooth* point to its tangent space, considered as a point in the Grassmannian $\text{Gr}(t, n)$.



Generic global rigidity, version 1

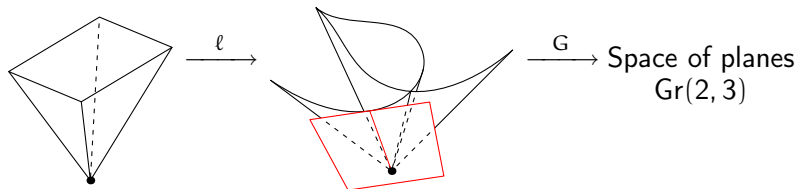
Theorem (Connelly \Rightarrow '95–05, Gortler-Healy-T \Leftarrow)

Rank of Gauss map on M is $vd - d(d + 1)$

\Leftrightarrow *graph is generically globally rigid*

Interpretation:

- ▶ Group $\text{Aff}(d)$ of affine transformations has dimension $d(d + 1)$
- ▶ We will see kernel of $(G \circ \ell)^*$ contains tangent to $\text{Aff}(d)$
- ▶ Graph is generically globally rigid iff kernel is no bigger

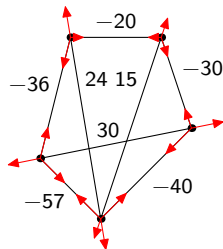


Stress vectors

Definition

A *stress vector* of a framework ρ is a function ω on the edges of ρ so that:

- ▶ $\forall \mathbf{u} : \rho(\mathbf{u}) = \frac{\sum_w \omega(\mathbf{u}, w) \rho(w)}{\sum_w \omega(\mathbf{u}, w)}$
(Each vertex is weighted avg of neighbors)
 - ▶ $\forall \mathbf{u} : \sum_w \omega(\mathbf{u}, w) (\rho(\mathbf{u}) - \rho(w)) = 0$
 - ▶ “Spring weights” balance out to leave framework in equilibrium
 - ▶ Vector is perpendicular to the span of ℓ^*
- (All conditions equivalent.)



Stresses: Easy facts

Stress vector: $\forall \mathbf{u} : \sum_{\mathbf{w}} \omega(\mathbf{u}, \mathbf{w})(\rho(\mathbf{u}) - \rho(\mathbf{w})) = 0$

Condition is linear in ω with ρ fixed

\Rightarrow Set of stresses for a given ρ is vector space

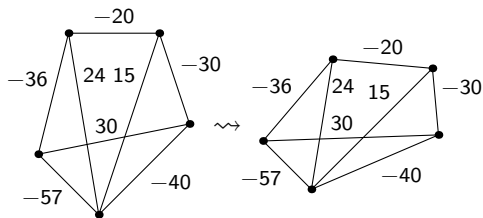
Condition is linear in ρ with ω fixed

\Rightarrow If ρ satisfies ω , so does any affine transform of ρ

\Rightarrow Whether ρ satisfies ω only depends on coord projections

Let $K(\rho)$ be 1D frameworks that satisfy all stresses that ρ satisfies.

$\dim K(\rho) \geq d + 1$ for non-flat ρ , but may be larger



Generic global rigidity, version 2

Theorem (Connelly \Rightarrow '95–05, Gortler-Healy-T \Leftarrow)

$\dim K(\rho) = d + 1$ for a generic ρ
 \Leftrightarrow graph is generically globally rigid

Proof (\Rightarrow , sketch).

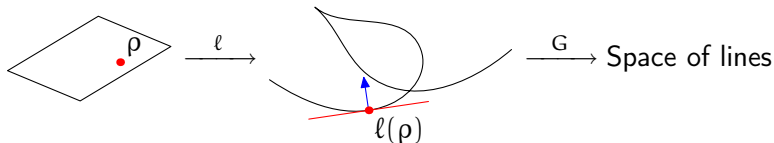
$\dim K(\rho) = d + 1$

\Leftrightarrow only affine images of ρ have all the same stresses as ρ

\Rightarrow generically, only affine images can have same tangent space in M

\Rightarrow generically, only affine images can map to same point in M

... only isometric images can map to same point. □



Equivalence of statements

Theorem

Rank of Gauss map on measurement set M is $vd - d(d + 1)$

\Leftrightarrow graph is generically globally rigid

Theorem

$\dim K(\rho) = d + 1$ for generic ρ

\Leftrightarrow graph is generically globally rigid

Lemma

Rank of Gauss map on M is $vd - d(\dim K(\rho))$.

Proof.

$A(\rho) :=$ d -dim frameworks that satisfy all stresses ρ satisfies $= K(\rho)^d$.

Fiber of $G \circ \ell$ is contained in $A(\rho)$ as an open subest. □

Proof idea

Theorem

$\dim K(\rho) > d + 1$ for generic ρ
 \Rightarrow graph is not generically globally rigid

Proof idea.

Given generic framework ρ , $\dim K(\rho) > d + 1$:

- ▶ construct version of ℓ between two spaces: $f : X \rightarrow Y$;
- ▶ degree (mod two) is defined;
- ▶ degree is zero;
- ▶ alternate preimage of ρ is global flex.



Degrees

Theorem

Given $f : X \rightarrow Y$, where

- ▶ X, Y manifolds of same dimension,
- ▶ ~~X compact~~, f proper,
- ▶ Y connected.

Then there is a mod-two $\deg f$. $|f^{-1}(y)| \equiv \deg f$ when y is regular value.

Can allow X to have singularities of codimension at least 2:

- ▶ remove image of singularities from Y
- ▶ remove preimage of image from X
- ▶ f is still proper, Y is still connected

The domain

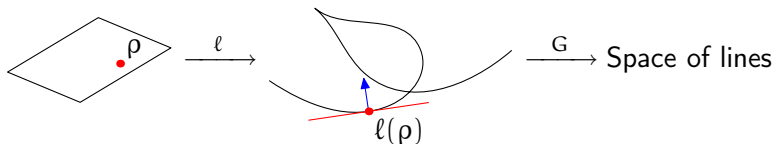
Recall/define:

- ▶ $K(\rho) =$ 1-dim frameworks satisfying all stresses that ρ satisfies
- ▶ $A(\rho) =$ d-dim frameworks satisfying all stresses that ρ satisfies
 $= K(\rho)^d \supset G^{-1}(\ell^{-1}(G(\ell(\rho))))$

Domain X is $A(\rho)/\text{Eucl}(d)$.

Lemma

If $\dim K(\rho) > d + 1$, singularities of $A(\rho)/\text{Eucl}(d)$ have codim at least 2.



The range

Image of $\mathbb{A}(\rho) \approx$ fiber of G through $\ell(\rho)$

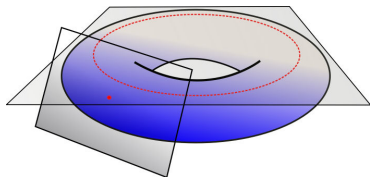
Gauss map is not arbitrary!

Theorem

*For an irreducible projective variety,
generic fibers of Gauss map are linear.*

Our measurement set M is semi-algebraic set
 \Rightarrow fibers of Gauss map are generically open
subsets of linear spaces.

Range Y is linear space $L(\rho)$ containing
fiber of Gauss map through $\ell(\rho)$.



The map

Recall: wanted $f : X \rightarrow Y$, with $\dim X = \dim Y$, f proper, Y connected, singularities of X of codimension 2.

Assume graph is generically locally rigid.

Map f is restriction of ℓ to $A(\rho)/\text{Eucl}(d) \rightarrow L(\rho)$.

Properties:

- ▶ f is proper
- ▶ $A(\rho)/\text{Eucl}(d)$ and $L(\rho)$ have same dimension (local rigidity)
- ▶ $A(\rho)/\text{Eucl}(d)$ has singularities of codim at least 2
- ▶ f is not onto (edge length² is positive)
- ▶ ρ is a regular point of f (local rigidity or genericity)

Thus $\deg f = 0$ and there is an alternate framework with same edge lengths as ρ . □

More algebraic?

Let $M_k =$ image of ℓ on k -dim frameworks

$$M_d = \underbrace{M_1 + \cdots + M_1}_{d \text{ copies}} = \text{sec}^d(M_1)$$

Can we prove a similar theorem with

- ▶ more general quadratic map ℓ ?
- ▶ over a field other than \mathbb{R} ?
- ▶ with a different signature of metric on \mathbb{R}^d ?

Proof does not generalize: used the fact that edge lengths are positive

More combinatorial?

The condition is efficiently checkable, but with a probabilistic algorithm.

Can we find

- ▶ a deterministic, polynomial-time algorithm?
- ▶ a more combinatorial description? (Yes in 2D)
- ▶ more examples?
Hendrickson found easier necessary conditions (HC) for global rigidity.
Only one known family of examples where HC not sufficient.

Changing dimension

Theorem (Gortler-Healy-T)

Let ρ be generic, locally but not globally rigid framework in \mathbb{E}^d .
Then $\exists \rho'$ so ρ can be connected to ρ' by path in \mathbb{E}^{d+1} .

Definition

A framework is *universally rigid* if every other framework with same edge lengths *in any dimension* is related by an isometry.

The vertex positions of a universally rigid framework can be found with semi-definite programming. (Good for applications.)

For which graphs is every generic framework in \mathbb{E}^d universally rigid?