Characterizing generic global rigidity

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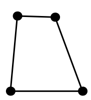
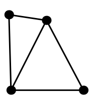
Flavors of rigidity

- A *framework* in $\mathbb{E}^d$ is a graph and a map from its vertices to $\mathbb{E}^d$.
- A framework is *locally rigid in* $\mathbb{E}^d$ if every other framework in a small neighborhood with same edge lengths is related to it by an isometry of $\mathbb{E}^d$.
- A framework is *globally rigid in* $\mathbb{E}^d$ if every other framework in $\mathbb{E}^d$ with same edge lengths is related to it by an isometry of $\mathbb{E}^d$.

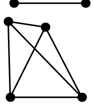
2D Examples



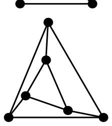
Not locally rigid



Locally but not globally rigid



Globally rigid



Locally but not globally rigid

Aside: Simplices

Theorem (Asimow-Roth '78)

Any framework whose graph is a complete graph is globally rigid.

A framework with $d + 1$ or fewer vertices in d dimensions is locally rigid iff graph is complete.

Thus no interesting questions for graphs with few vertices.
Will assume graphs have at least $d + 2$ vertices.

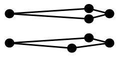
Rigidity is NP-hard...

Theorem (Saxe '79)

Checking whether a framework with integer coordinates is globally rigid is NP-hard.

Idea: In 1D, need to solve a partition problem.

Locally but not globally rigid (1D)



Globally rigid (1D)

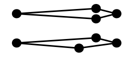
So problem seems hopeless!

...Generic rigidity is easy

Definition

A framework is *generic* if its coordinates do not satisfy any polynomial equation.

Non-generic



Generic

Definition

A graph is *generically globally rigid* if a generic framework of it is globally rigid.

We

- characterize generically globally rigid graphs;
- show global rigidity is independent of the generic framework; and
- give an efficient randomized algorithm for checking the condition.

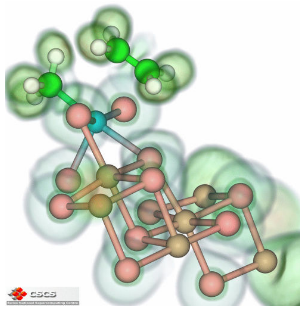
History and applications

2D case understood completely (Laman '70, Lovász-Yemini '82, Jackson-Jordán '05).

Hendrickson '92 gave simple necessary conditions for global rigidity; Connelly showed these were not sufficient in 3D.

People care!
Reconstruction: given some distances between nodes, reconstruct framework.
Global rigidity necessary to be well-posed.
Applications:

- molecular chemistry
- sensor networks



Geometry of maps

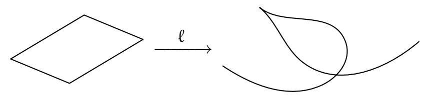
Definition

The *length-squared function* ℓ is the map from frameworks of a graph to its edge lengths, squared.

Definition

The *rigidity matrix* of a framework is the Jacobian ℓ^* of ℓ.

The *rank* of an algebraic map is the rank of its linearization at generic points (= dimension of the image).



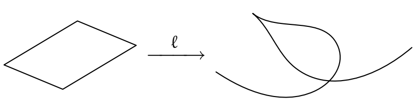
Generic local rigidity

Theorem (Asimow-Roth '78)

A graph is generically locally rigid ⇔ *rank of* ℓ^* *is generically* $vd - \frac{d(d+1)}{2}$.

Interpretation:

- Group $\mathrm{Eucl}(d)$ of isometries has dimension $\frac{d(d+1)}{2}$.
- Kernel of ℓ^* always contains tangent to $\mathrm{Eucl}(d)$.
- Graph is generically locally rigid iff $\ker \ell^*$ no bigger.

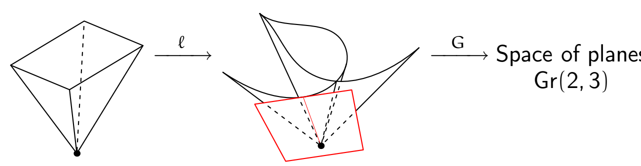


The Gauss map

Definition

The *measurement set* M is the image of ℓ.

The *Gauss map* G of a homogeneous semi-algebraic set of dimension t in $\mathbb{R}^n$ takes each smooth point to its tangent space, considered as a point in the Grassmannian $\mathrm{Gr}(t, n)$.



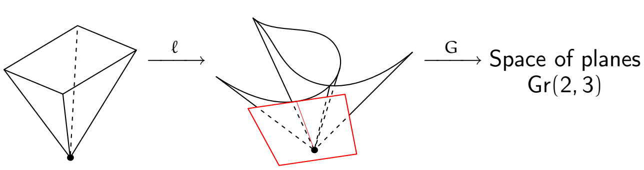
Generic global rigidity, version 1

Theorem (Connelly ⇒ '95–05, Gortler-Healy-T ⇐)

Rank of Gauss map on M *is* $vd - d(d+1)$
⇔ *graph is generically globally rigid*

Interpretation:

- Group $\mathrm{Aff}(d)$ of affine transformations has dimension $d(d+1)$
- We will see kernel of $(G \circ \ell)^*$ contains tangent to $\mathrm{Aff}(d)$
- Graph is generically globally rigid iff kernel is no bigger



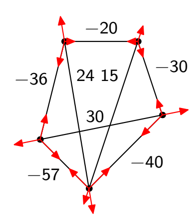
Stress vectors

Definition

A *stress vector* of a framework ρ is a function ω on the edges of ρ so that:

- $\forall u : \rho(u) = \frac{\sum_w \omega(u,w)\rho(w)}{\sum_w \omega(u,w)}$
  (Each vertex is weighted avg of neighbors)
- $\forall u : \sum_w \omega(u,w)(\rho(u) - \rho(w)) = 0$
- "Spring weights" balance out to leave framework in equilibrium
- Vector is perpendicular to the span of ℓ^*

(All conditions equivalent.)



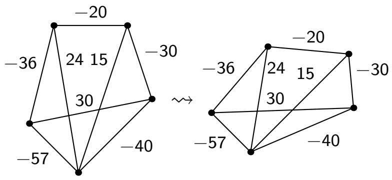
Stresses: Easy facts

Stress vector: $\forall u : \sum_{w} \omega(u, w)(\rho(u) - \rho(w)) = 0$

Condition is linear in ω with ρ fixed
$\Rightarrow$ Set of stresses for a given ρ is vector space

Condition is linear in ρ with ω fixed
$\Rightarrow$ If ρ satisfies ω, so does any affine transform of ρ
$\Rightarrow$ Whether ρ satisfies ω only depends on coord projections

Let $K(\rho)$ be 1D frameworks that satisfy all stresses that ρ satisfies.
$\dim K(\rho) \geqslant d + 1$ for non-flat ρ, but may be larger



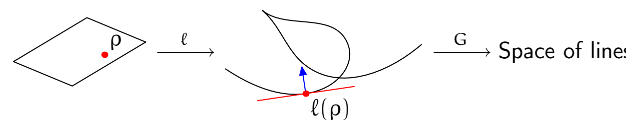
Generic global rigidity, version 2

Theorem (Connelly ⇒ '95–05, Gortler-Healy-T ⇐)

$\dim K(\rho) = d + 1$ *for a generic* ρ
$\Leftrightarrow$ *graph is generically globally rigid*

Proof (⇒, sketch).

$\dim K(\rho) = d + 1$
$\Leftrightarrow$ only affine images of ρ have all the same stresses as ρ
$\Rightarrow$ generically, only affine images can have same tangent space in M
$\Rightarrow$ generically, only affine images can map to same point in M
. . . only isometric images can map to same point. □



Equivalence of statements

Theorem

Rank of Gauss map on measurement set M *is* $vd - d(d + 1)$
$\Leftrightarrow$ *graph is generically globally rigid*

Theorem

$\dim K(\rho) = d + 1$ *for generic* ρ
$\Leftrightarrow$ *graph is generically globally rigid*

Lemma

Rank of Gauss map on M *is* $vd - d(\dim K(\rho))$.

Proof.

$A(\rho) :=$ d-dim frameworks that satisfy all stresses ρ satisfies $= K(\rho)^d$.
Fiber of $G \circ \ell$ is contained in $A(\rho)$ as an open subest. □

Proof idea

Theorem

$\dim K(\rho) > d + 1$ *for generic* ρ
$\Rightarrow$ *graph is not generically globally rigid*

Proof idea.

Given generic framework ρ, $\dim K(\rho) > d + 1$:

- construct version of ℓ between two spaces: $f : X \to Y$;
- degree (mod two) is defined;
- degree is zero;
- alternate preimage of ρ is global flex. □

Degrees

Theorem

Given $f : X \to Y$, *where*

- X, Y *manifolds of same dimension,*
- ~~*X compact,*~~ f *proper,*
- Y *connected.*

Then there is a mod-two deg f. $|f^{-1}(y)| \equiv \deg f$ *when* y *is regular value.*

Can allow X to have singularities of codimension at least 2:

- remove image of singularities from Y
- remove preimage of image from X
- f is still proper, Y is still connected

The domain

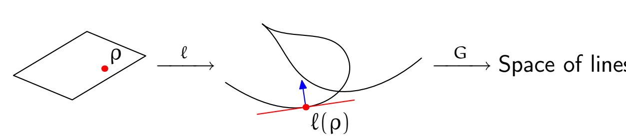
Recall/define:

- $K(\rho) =$ 1-dim frameworks satisfying all stresses that ρ satisfies
- $A(\rho) =$ d-dim frameworks satisfying all stresses that ρ satisfies
  $= K(\rho)^d \supset G^{-1}(\ell^{-1}(G(\ell(\rho))))$

Domain X is $A(\rho)/\,\mathrm{Eucl}(d)$.

Lemma

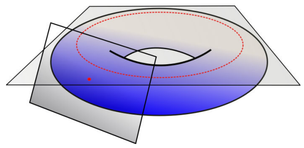
If $\dim K(\rho) > d + 1$, *singularities of* $A(\rho)/\,\mathrm{Eucl}(d)$ *have codim at least 2.*



The range

Image of $A(\rho) \approx$ fiber of G through $\ell(\rho)$

Gauss map is not arbitrary!



Theorem

For an irreducible projective variety, generic fibers of Gauss map are linear.

Our measurement set M is semi-algebraic set $\Rightarrow$ fibers of Gauss map are generically open subsets of linear spaces.

Range Y is linear space $L(\rho)$ containing fiber of Gauss map through $\ell(\rho)$.

The map

Recall: wanted $f : X \to Y$, with $\dim X = \dim Y$, f proper, Y connected, singularities of X of codimension 2.

Assume graph is generically locally rigid.

Map f is restriction of ℓ to $A(\rho)/\operatorname{Eucl}(d) \to L(\rho)$.

Properties:

- f is proper
- $A(\rho)/\operatorname{Eucl}(d)$ and $L(\rho)$ have same dimension (local rigidity)
- $A(\rho)/\operatorname{Eucl}(d)$ has singularities of codim at least 2
- f is not onto (edge length^2 is positive)
- ρ is a regular point of f (local rigidity or genericity)

Thus $\deg f = 0$ and there is an alternate framework with same edge lengths as ρ. □

More algebraic?

Let M_k = image of ℓ on k-dim frameworks

$$M_d = \underbrace{M_1 + \cdots + M_1}_{d \text{ copies}} = \sec^d(M_1)$$

Can we prove a similar theorem with

- more general quadratic map ℓ?
- over a field other than $\mathbb{R}$?
- with a different signature of metric on $\mathbb{R}^d$?

Proof does not generalize: used the fact that edge lengths are positive

More combinatorial?

The condition is efficiently checkable, but with a probabilistic algorithm.

Can we find

- a deterministic, polynomial-time algorithm?
- a more combinatorial description? (Yes in 2D)
- more examples?
  Hendrickson found easier necessary conditions (HC) for global rigidity.
  Only one known family of examples where HC not sufficient.

Changing dimension

Theorem (Gortler-Healy-T)

*Let ρ be generic, locally but not globally rigid framework in $\mathbb{E}^d$.
Then $\exists \rho'$ so ρ can be connected to ρ' by path in $\mathbb{E}^{d+1}$.*

Definition

A framework is *universally rigid* if every other framework with same edge lengths *in any dimension* is related by an isometry.

The vertex positions of a universally rigid framework can be found with semi-definite programming. (Good for applications.)
For which graphs is every generic framework in $\mathbb{E}^d$ universally rigid?