

Combinatorial link Floer homology

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<http://www.math.columbia.edu/~dpt/speaking>

October 11, 2010, Temple University

Outline

4-dimensional invariants

Several theories giving 4-manifold invariants:

Donaldson theory

\leftrightarrow (conj. Seiberg–Witten '94)

Monopoles (Seiberg–Witten)

\cong (Taubes '08)

Embedded contact homology (ECH)

\cong (Cutluhan-Lee-Taubes, Colin-Ghiggini-Honda '10)

Heegaard Floer (HF) homology

We topologists only have one trick in 4 dimensions!

Each theory has advantages.

Focus

\widehat{HF} homology, the most computable.

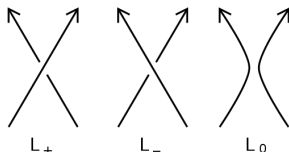
3-dimensional invariants

By contrast, there are many 3-manifold invariants.
Focus on a knot K in the space.

One of oldest: Alexander polynomial $\Delta(K)$.

Algebraic topology: Look at $H_1(\mathbb{R}^3 \setminus K)$ under deck transforms.

Skein theory: $\Delta(L_+) - \Delta(L_-) = (t^{1/2} - t^{-1/2})\Delta(L_0)$



More recent: Jones polynomial, HOMFLYPT polynomial, ...

How are 3- and 4-dimensional theories related?

Knot homologies

Many knot invariants are one- or two-variable Laurent polynomials.

Can often find a doubly- or triply-graded homology theory whose Euler characteristic is the polynomial invariant.

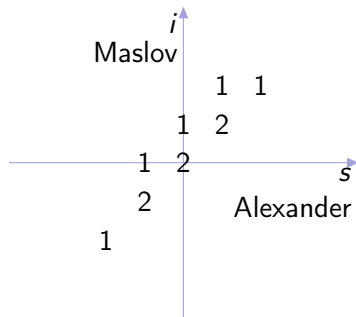
Knot polynomial	Knot homology
Alexander (1928)	{ Heegaard Floer (2002) Instanton Floer (2008)
Jones (1983)	Khovanov (1999)
HOMFLYPT (1985)	Khovanov-Rozansky (2004)
Kauffman (1990)	Khovanov-Rozansky (2007) (conj)

Passage polynomial \Rightarrow homology called *categorification*.

Heegaard Floer homology

$$\dim(\widehat{HFK}_i(K; s)):$$

($K = 10_{132}$)



Characteristics of \widehat{HFK} :

- ▶ **Bigraded;**
- ▶ Euler characteristic is Conway-Alexander polynomial Δ ;
- ▶ Max grading is knot genus (so detects unknot); (Ozsváth-Szabó 2001)
- ▶ Determines knot fibration; (Ghiggini, Ni 2006)
- ▶ Gives a new and effective transverse knot invariant;
- ▶ Defined via pseudo-holomorphic curves.

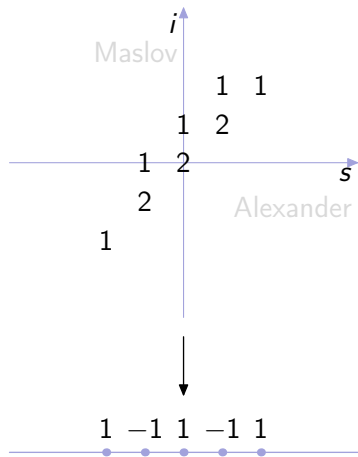
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...and so the world's simplest algorithm for knot genus!

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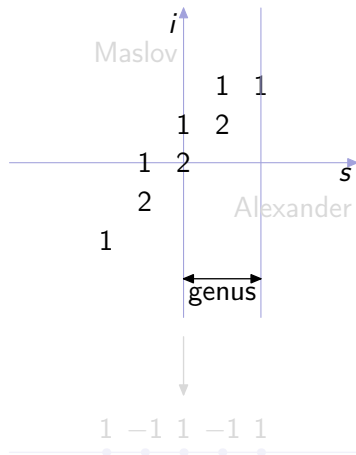
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Knot genus

Definition

A *Seifert surface* for a knot is an oriented surface embedded in space whose boundary is the knot.

The *genus* of a knot is the minimal genus of any Seifert surface.

Seifert surfaces always exist.

The genus of a knot is 0 iff it is the unknot.



Theorem (Neuwirth 1960)

Genus of $K \geq$ degree of Alexander-Conway polynomial

Theorem (Ozsváth-Szabó 2001)

Genus of $K = \max s$ so that $\widehat{HFK}_(K; s) \neq 0$*

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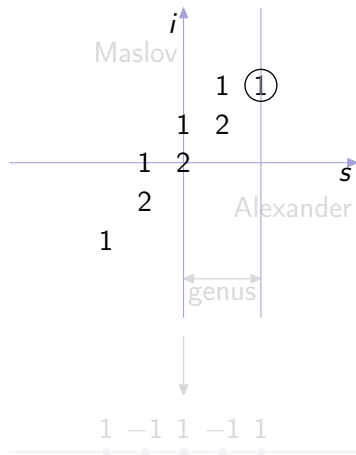
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Fibered knots

Definition

A knot is *fibered* if complement is a fiber bundle of a surface over the circle.

Or: Seifert surface can be swept around to cover complement.

Theorem (Neuwirth 1960)

K is fibered \Rightarrow Alexander-Conway polynomial is monic

Theorem (Ghiggini-Ni 2006)

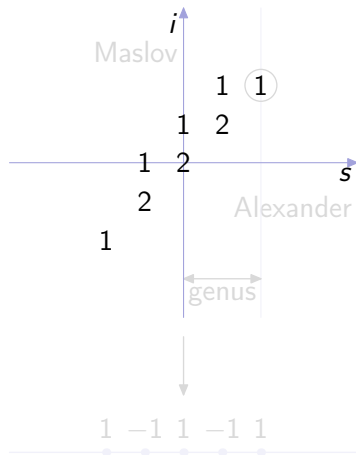
K is fibered $\Leftrightarrow \widehat{HFK}_i(K; s)$ is monic w.r.t. s

\Leftrightarrow for max s so that $\widehat{HFK}_*(K; s) \neq 0$, $\dim(\widehat{HFK}_*(K; s)) = 1$

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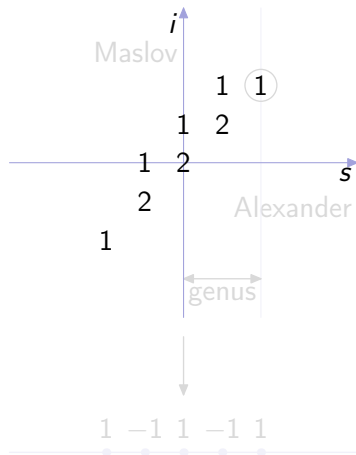
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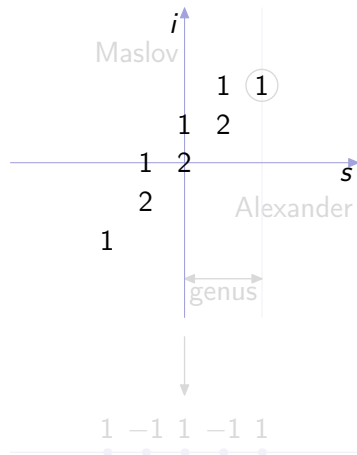
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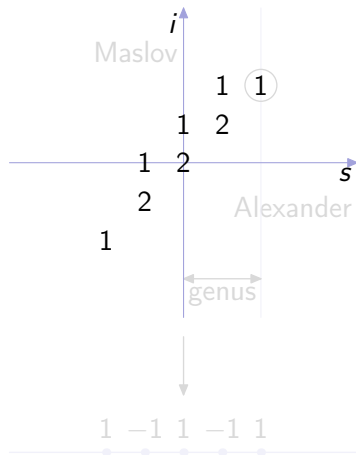
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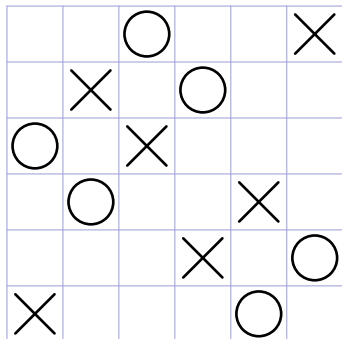
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Setting: Grid diagrams

Grid diagram: square diagram with one X and one O per row and column.



Turn it into a knot: connect
 X to O in each column;
 O to X in each row.

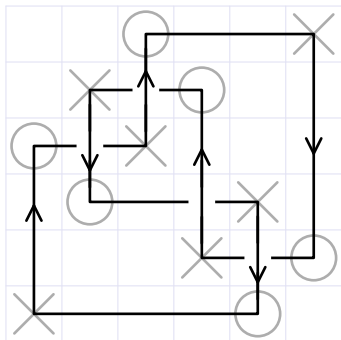
Cross vertical strands over horizontal.

Grid diagrams exist: take any diagram,
rotate crossings so vertical crosses over
horizontal.

The knot is unchanged under
cyclic rotations:

Move top segment to bottom.

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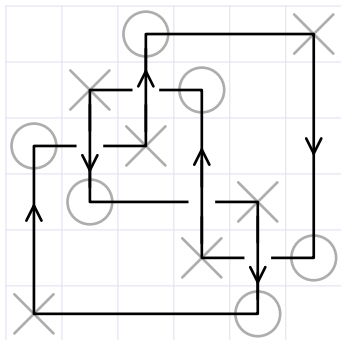
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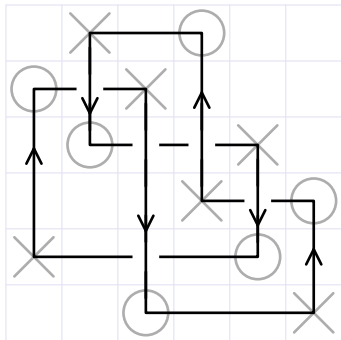
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Computing the Alexander polynomial

We categorify the following formula:

$$\begin{vmatrix}
 1 & 1 & 1 & t & t & t \\
 1 & 1 & t^{-1} & 1 & t & t \\
 1 & t & 1 & 1 & t & t \\
 1 & t & t & t & t^2 & t \\
 1 & t & t & t & t & 1 \\
 1 & 1 & 1 & 1 & 1 & 1
 \end{vmatrix}
 = \pm t^*(1-t)^{n-1} \Delta(K; t)$$

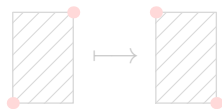
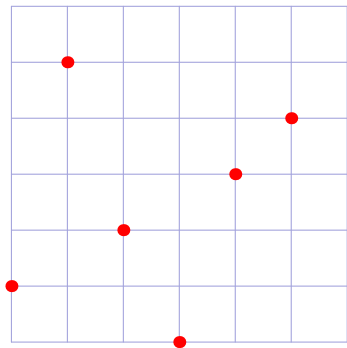
- ▶ Make matrix of $t^{-\text{winding \#}}$
(with extra row/column of 1's);
- ▶ det determines the Conway-Alexander polynomial Δ
($n = \text{size of diagram; here } 6$)

Outline

Computing $HF\tilde{K}$: Chain complex $\tilde{C}K$

Define a chain complex $\tilde{C}K$ over \mathbb{F}_2 .

- ▶ $n!$ generators: matchings between horizontal and vertical gridcircles (as counted in \det for Alexander).
- ▶ Boundary ∂ switches corners on *empty rectangles*:

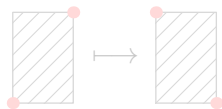
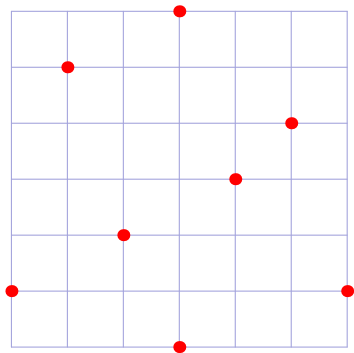


Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no X 's, O 's, or other points in generator.)

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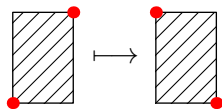
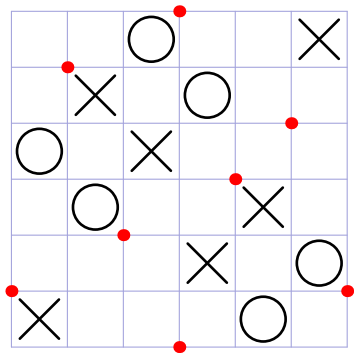


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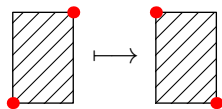
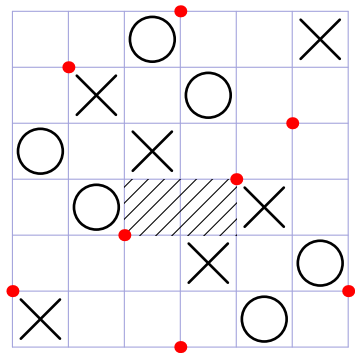


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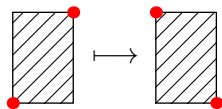
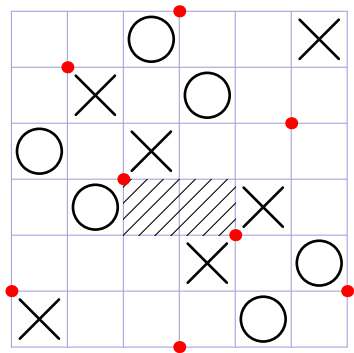


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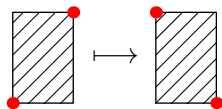
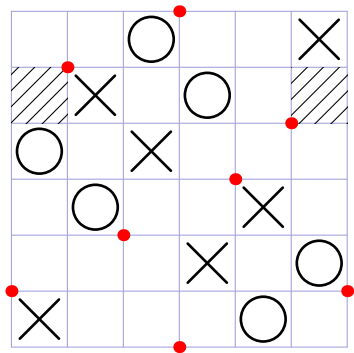


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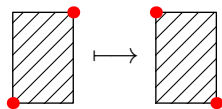
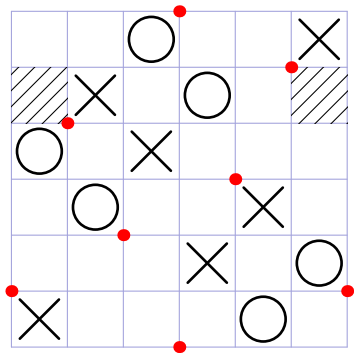


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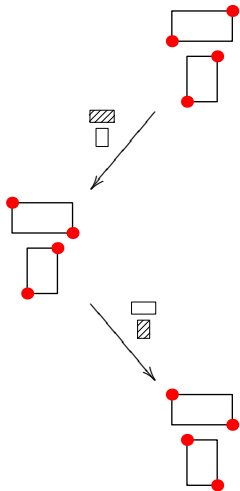
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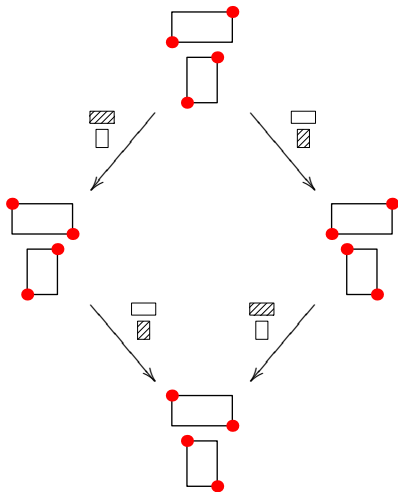
Computing HFK : $\partial^2 = 0$



Each term in ∂^2 must have a mate:

- ▶ If rectangles are disjoint, take rectangles in either order.
- ▶ If rectangles share a corner, decompose the union in another way.

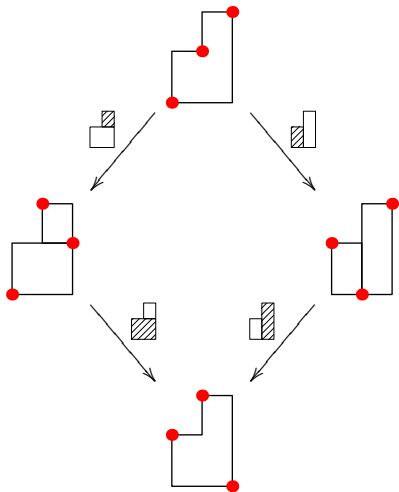
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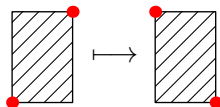


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Computing HFK : Gradings on \widetilde{CK}

In the plane,



removes one *inversion*.

For $A, B, C \subset \mathbb{R}^2$,

$$\begin{aligned} \mathcal{I}(A, B) &:= \#\{ a \square b \mid a \in A, b \in B \} \\ \mathcal{I}(A - B, C) &:= \mathcal{I}(A, C) - \mathcal{I}(B, C) \end{aligned}$$

For \mathbf{x} a generator, \mathbb{X} = set of X 's, \mathbb{O} = set of O 's, gradings are:

- ▶ **Maslov:** $M(\mathbf{x}) := \mathcal{I}(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) + 1$.
- ▶ **Alexander:** Sum of winding numbers around generator pts, or $A(\mathbf{x}) := \frac{1}{2}(\mathcal{I}(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) - \mathcal{I}(\mathbf{x} - \mathbb{X}, \mathbf{x} - \mathbb{X}) - (n - 1))$.

Computing *HFK*: The answer

Theorem (Manolescu-Ozsváth-Sarkar)

For G a grid diagram for K ,

$$H_*(\widehat{CK}(G)) \simeq \widehat{HFK}(K) \otimes V^{\otimes n-1}$$

where $V := (\mathbb{F}_2)_{0,0} \oplus (\mathbb{F}_2)_{-1,-1}$.

(Remember the factor of $(1-t)^{n-1}$ in determinant formula for Δ .)

Gillam and Baldwin used this to compute \widehat{HFK} for all knots with ≤ 11 crossings, including new values of knot genus.

Variants

- ▶ Allow rectangles to cross O 's with a coefficient of U_i
 - ▶ Remove factor of $V^{\otimes n-1}$
 - ▶ Get stronger invariant HFK^- over $\mathbb{F}_2[U]$
- ▶ Allow rectangles to cross X 's to get a filtered complex
- ▶ Add signs (in essentially unique way) to work over $\mathbb{Z}[U]$.

		Cross X 's?	
		No	Yes
Cross O 's? (count with U_i variables)	No	$\widehat{HFK} \otimes V^{n-1}$	Filtered version
	All but 1	\widehat{HFK}	"
	Yes	HFK^-	"

Outline

TQFT-like structure

Geometry	Algebra
Closed 4-manifold W^4 , Spin ^c structure \mathfrak{s}	Invariant $HF(W, \mathfrak{s})$
3-manifold Y^3 , Spin ^c structure \mathfrak{s}	Homology theory $HF(Y, \mathfrak{s})$
Cobordism $\partial W^4 = (-Y_1) \cup Y_2$	$HF(W) : HF(Y_1) \rightarrow HF(Y_2)$

Subtleties omitted here. E.g., if you make an ordinary TQFT, invariants of closed 4-manifolds are 0.

Similar story for knots in 3-manifolds with surface cobordisms.

We saw $HFK(S^3, K; \mathbb{F}_2)$, the easiest to understand.

Extending TQFT down

Geometry	Algebra
Closed 4-manifold W^4	Invariant $HF(W)$
3-manifold Y^3	Homology $HF(Y)$
Surface F	Algebra $\mathcal{A}(F)$
3-manifold w/ $\partial Y = F$	\mathcal{A}_∞ module $\widehat{CF}(Y)$
Gluing $Y = Y_1 \cup_F Y_2$	$\widehat{CF}(Y) \simeq \widehat{CF}(Y_1) \tilde{\otimes}_{\mathcal{A}(F)} \widehat{CF}(Y_2)$
3-manifold w/ $\partial Y^3 = F_1 \cup F_2$	Bimodules $\widehat{CF}(Y), \dots$
Cobordism $\partial W^4 = -Y_1 \cup Y_2$	Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \rightarrow \widehat{HF}(Y_2)$
Self-gluing	Hochschild (co)homology

This is the theory of Bordered Floer homology.

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Cobordism $\partial W^4 = -Y_1 \cup Y_2$	Map $\widehat{HF}(W) : \widehat{HF}(Y_1) \rightarrow \widehat{HF}(Y_2)$
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Extending TQFT down

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Mapping class group

Special case: 3-manifold is $F \times [0, 1]$, with two boundary components parametrized differently.

This corresponds to element of *mapping class group*: diffeomorphisms $\phi : F \rightarrow F$, considered up to isotopy.

Theorem (Lipshitz-Ozsváth-T)

There is an action of the strongly based mapping class group on modules over $\mathcal{A}(F)$.

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This action is faithful.

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Special case: 3-manifold is $F \times [0, 1]$, with two boundary components parametrized differently.

This corresponds to element of **strongly based mapping class group**: diffeomorphisms $\phi : F \rightarrow F$ **fixing a disk**, considered up to isotopy **fixing a disk**.

Theorem (Lipshitz-Ozsváth-T)

There is an action of the strongly based mapping class group on modules over $\mathcal{A}(F)$.

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This action is faithful.

Outline

Further questions

There are many questions left!

- ▶ Combinatorial proofs of genus equality and other nice properties?
- ▶ Construct a topological space with this homology?
- ▶ Manifold version of other link homologies?
- ▶ Make entire theory computable?
- ▶ Do computations!

Appendix: Crossing number vs. Grid number

Knots are usually ordered by *crossing number*:

Minimum number of crossings in a planar diagram.

For grid diagrams, natural to consider *grid number* (or *arc index*):

Minimum size of a grid diagram.

Theorem (Bae–Park, Morton–Beltrami)

Grid number of an alternating knot is equal to crossing number + 2.

For non-alternating knots, grid number strictly less.