Combinatorial link Floer homology and transverse knots

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What are knot homologies?

Many knot invariants are one- or two-variable Laurent polynomials, associated to quantum groups.

Can often find a doubly- or triply-graded homology theory whose Euler characteristic is the polynomial invariant.

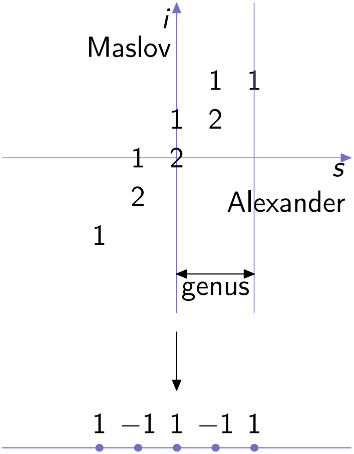
Group	Knot poly	Knot homology
$SL(2)$	Jones $J(t)$	Khovanov (1999)
$SL(n)$	HOMFLY $H(a, z)$	Kh-Roz (2004) ($n \in \mathbb{Z}$)
		Kh-Roz (2005) (n variable)
$GL(1 \mid 1)$	Alexander $\Delta(t)$	Heegaard Floer
		Seiberg–Witten Floer
$OSp(n)$	Kauffman $F(a, z)$	Kh-Roz (2007) (conjectural)

Passage polynomial $\Rightarrow$ homology called *categorification*.

Similar picture for 3-manifolds for Heegaard Floer homology.

What is Heegaard Floer homology?

$\dim(\widehat{HFK}_i(K;s))$: $(K = 10_{132})$



Characteristics of $\widehat{HFK}$:

- Bigraded;
- Euler characteristic is Conway-Alexander polynomial;
- Max grading is knot genus; (Ozsváth-Szabó 2001)
- Determines knot fibration; (Ghiggini, Ni 2006)
- Gives new and effective transverse knot invariant;
- Defined via pseudo-holomorphic curves.

We will give a simple algorithm for computing HFK...

...and so the world's simplest algorithm for knot genus!

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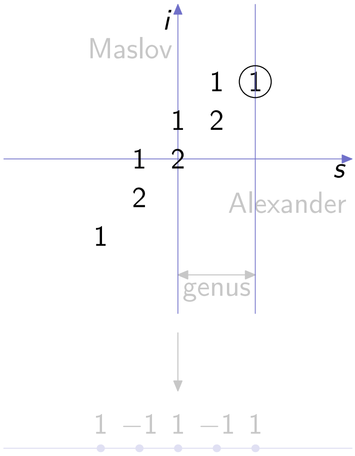

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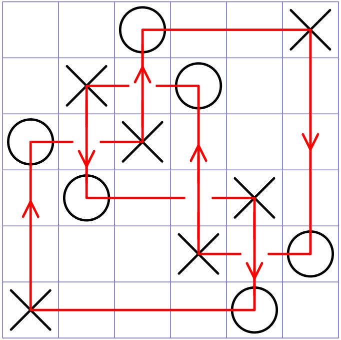
Setting: Grid diagrams



Grid diagram: square diagram with one X and one O per row and column.

Turn it into a knot: connect

X to O in each column;

O to X in each row.

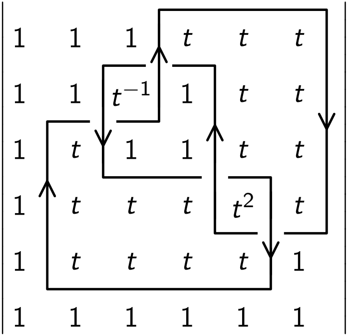
Cross vertical strands over horizontal.

Grid diagrams exist: take any diagram, rotate crossings so vertical crosses over horizontal.

The knot is unchanged under *cyclic rotations*:
Move top segment to bottom.

Computing the Alexander polynomial

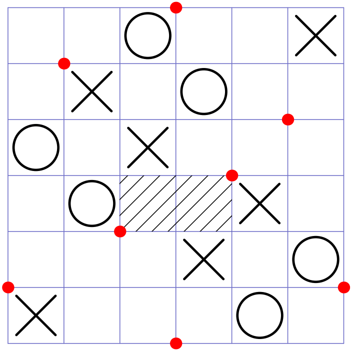
We categorify the following formula:



$$
\begin{vmatrix}
1 & 1 & 1 & t & t & t \\
1 & 1 & t^{-1} & 1 & t & t \\
1 & t & 1 & 1 & t & t \\
1 & t & t & t & t^2 & t \\
1 & t & t & t & t & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{vmatrix}
= \pm t^{*}(1-t)^{n-1}\Delta(K;t)
$$

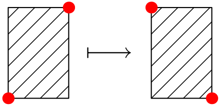
- Make matrix of $t^{-\text{winding } \#}$
  (with extra row/column of 1's);
- det determines the Conway-Alexander polynomial Δ
  ($n =$ size of diagram; here 6)

Computing *HFK*: Chain complex $\widetilde{CK}$



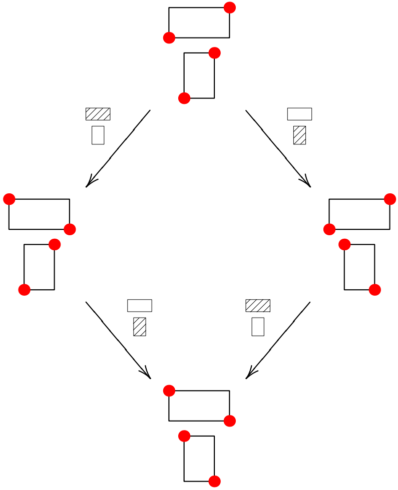
Define a chain complex $\widetilde{CK}$ over $\mathbb{F}_2$.

- $n!$ generators: matchings between horizontal and vertical gridcircles (as counted in det for Alexander).
- Boundary ∂ switches corners on *empty rectangles*:



Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no X's, O's, or other points in generator.)

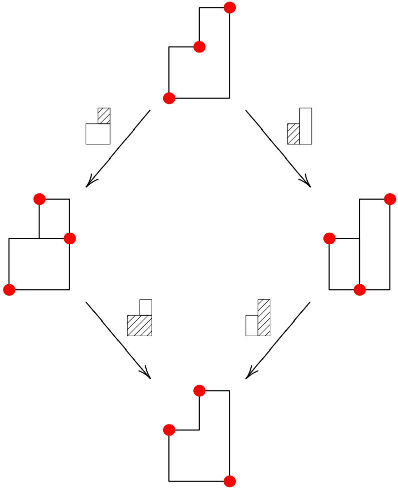
Computing *HFK*: $\partial^2 = 0$



Each term in ∂^2 must have a mate:

- If rectangles are disjoint, take rectangles in either order.
- If rectangles share a corner, decompose the union in another way.

Computing *HFK*: $\partial^2 = 0$



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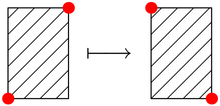
Computing HFK: Gradings on $\widetilde{CK}$

In the plane,



removes one *inversion*.

For $A, B, C \subset \mathbb{R}^2$,

$$\mathcal{I}(A,B) := \#\{\, {}_a\Box^b \mid a \in A, b \in B \,\}$$
$$\mathcal{I}(A-B,C) := \mathcal{I}(A,C) - \mathcal{I}(B,C)$$

For $\mathbf{x}$ a generator, $\mathbb{X}$ = set of X's, $\mathbb{O}$ = set of of O's, gradings are:

- **Maslov:** $M(\mathbf{x}) := \mathcal{I}(\mathbf{x}-\mathbb{O}, \mathbf{x}-\mathbb{O}) + 1$.
- **Alexander:** Sum of winding numbers around generator pts, or $A(\mathbf{x}) := \frac{1}{2}\big(\mathcal{I}(\mathbf{x}-\mathbb{O}, \mathbf{x}-\mathbb{O}) - \mathcal{I}(\mathbf{x}-\mathbb{X}, \mathbf{x}-\mathbb{X}) - (n-1)\big)$.

Computing *HFK*: The answer

Theorem (Manolescu-Ozsváth-Sarkar)

For G a grid diagram for K,

$$H_*(\widetilde{CK}(G)) \simeq \widehat{HFK}(K) \otimes V^{\otimes n-1}$$

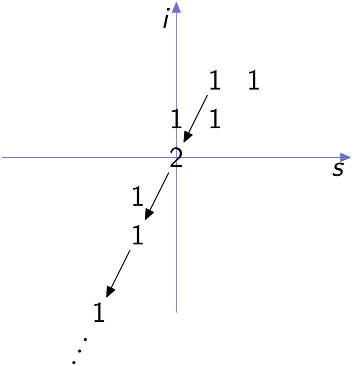
where $V := (\mathbb{F}_2)_{0,0} \oplus (\mathbb{F}_2)_{-1,-1}$.

(Remember the factor of $(1-t)^{n-1}$ in determinant formula for Δ.)

Gillam and Baldwin used this to compute $\widehat{HFK}$ for all knots with ≤ 11 crossings, including new values of knot genus.

Improving the answer

dim $HFK_i^-(K;s)$:



To remove factors of $V^{\otimes n-1}$:

Complex HFK^-: variant of $\widehat{HFK}$
Module over $\mathbb{F}_2[U]$
U has degree $(-1,-2)$
Related to $\widehat{HFK}$ by Universal Coefficient Theorem (set U to 0 on chains).

To compute: Add one U_i for each O.
Complex $CK^-(G)$ over $\mathbb{F}_2[U_1,\dots,U_n]$
∂ counts rects. that contain only O's, weighted by corresponding U_i.

Theorem (Manolescu-Ozsváth-Sarkar)

$H_*(CK^-(G)) \simeq HFK^-(K)$.
Each U_i acts by U on the homology.

Further variants

Can also:

- Allow rectangles to cross X's to get a filtered complex with trivial total homology.
- Add signs (in essentially unique way) to work over $\mathbb{Z}[U]$.

Cross X's?

Cross O's? (count with U_i variables)

	No	Yes
No	$\widehat{HFK} \otimes V^{n-1}$	Filtered version
All but 1	$\widehat{HFK}$	"
Yes	HFK^-	"

Combinatorial invariance

Theorem (Manolescu-Ozsváth-Szábo-T.)

For any sequence of elementary grid moves, there is an explicit chain map exhibiting invariance of HFK^-.

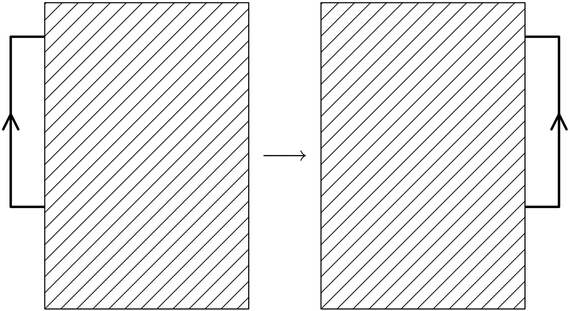
Theorem (Naturality or Functoriality, Ozsváth-Stipsicz 2008)

The chain map depends only on isotopy class of sequence of elementary grid moves. That is, oriented mapping class group of K *acts naturally on* $HFK^-(K)$.

Conjecture

Every cobordism between knots K_1 *and* K_2 *gives a filtered map between* $HFK^-(K_1)$ *and* $HFK^-(K_2)$.

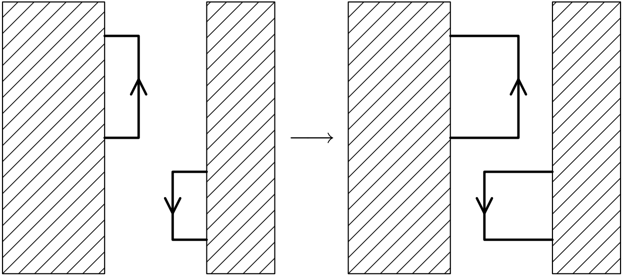
Elementary grid moves



- **Cycle:** Move left column to right, or top row to bottom.
- **Commute:** Switch two non-interfering columns or rows.
- **Stabilize:** Introduce a notch at a corner.

(Cromwell '95, Dynnikov '06)

Elementary grid moves



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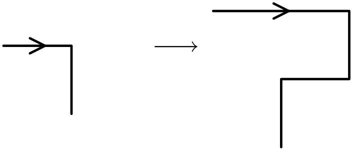

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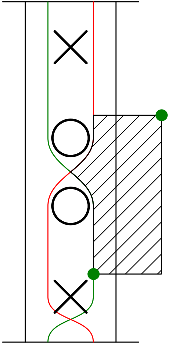
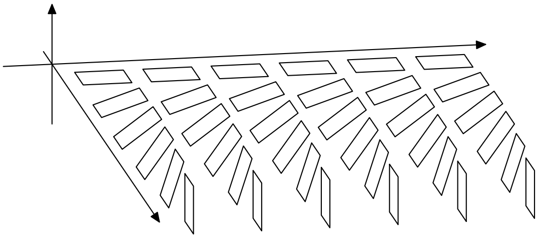
Chain map for commutation counts pentagons



To construct a chain map for commutation, draw two versions of the middle gridcircle on a single diagram.

The chain map counts empty pentagons going between the two gridcircles.

Contact structures and knots



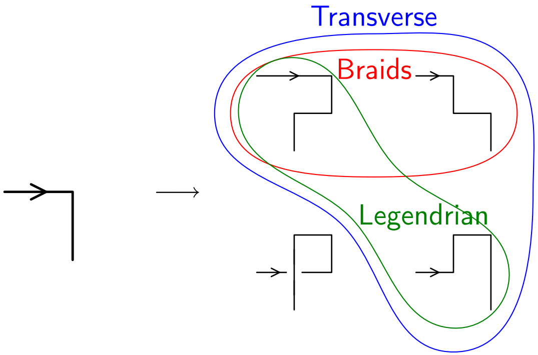
A *contact structure* is a twisted 2-plane field:
if α is a 1-form defining the plane field, $\alpha \wedge d\alpha$ is positive.
(Warning: above contact structure is reversed.)

A *Legendrian knot* is a knot that is tangent to the plane field.
A *transverse knot* is a knot that is transverse to the plane field.

Transverse knots have one easy invariant, the *self-linking number*.

Problem. Find transverse knots with the same classical knot type and self-linking number.

Ways to stabilize

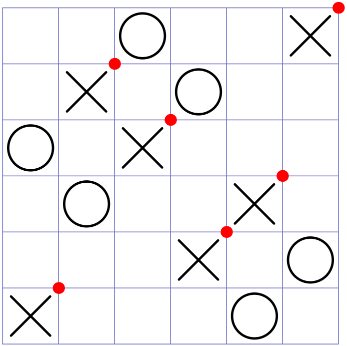


Four ways to stabilize: Where to leave the empty square?

- Two diagonal opposite ways preserve Legendrian knot.
- Two adjacent ways preserve closed braid mod exchange move.
- Three ways preserve transverse knot.

Warning: The Legendrian/transverse knots are mirrored.

Transverse invariant: Definition



Definition

The *canonical generator* $\mathbf{x}^+(G)$ is given by the upper-right corner of each X.

Facts:

- $\partial \mathbf{x}^+ = 0$. (The X's block any rectangles.)
- $[\mathbf{x}^+(G)]$ maps to $[\mathbf{x}^+(G')]$ under commutation and 3 out of 4 stabilizations.

Theorem (Ozsváth-Szabó-T.)

$[\mathbf{x}^+(G)]$ *in* $HFK^-(m(K))$ *is an invariant of the transverse knot represented by* G*, up to quasi-isomorphism of filtered complexes.*

Transverse invariant: Properties

Let G be a grid diagram representing the transverse knot $\mathcal{T}$.

- $\mathbf{x}^+(G)$ lives in bigrading $(s, 2s)$, where $s = \frac{sl(\mathcal{T})+1}{2}$.
- If $\mathcal{T}'$ differs from $\mathcal{T}$ by a positive stabilization, then $[\mathbf{x}^+(\mathcal{T}')] = U[\mathbf{x}^+(\mathcal{T})]$.
- $[\mathbf{x}^+(\mathcal{T})] \neq 0$ in HFK^-.

Corollary

For any transverse knot $\mathcal{T}$ of topological type K,

$$\frac{sl(\mathcal{T})+1}{2} \leq \tau(K)$$

where $\tau(K)$ is the largest Alexander grading which has an element which is not U torsion.

Ozsváth-Szabó 2003: $\tau(K)$ is less than the 4-ball genus.

Transverse invariant: Examples

Let $\theta(\mathcal{T})$ (resp. $\widehat{\theta}(\mathcal{T})$) be the transverse invariant in $HFK^{-}(m(K))$ (resp. $\widehat{HFK}(m(K))$).
$\widehat{\theta}(\mathcal{T}) = 0$ iff $\theta(\mathcal{T})$ is divisible by U.

Theorem (Ng-Ozsváth-T.)

The knots $m(10_{132})$ and $m(12n_{200})$ have two transverse representatives with the same sl, one with $\widehat{\theta} = 0$ and one with $\widehat{\theta} \neq 0$.

This technique also works for the $(2,3)$ cable of the $(2,3)$ torus knot, originally found by Etnyre-Honda and Menasco-Matsuda.

Transverse invariant: More examples

Let δ_1 be the next differential in the spectral sequence on $\widehat{HFK}$ converging to HFK^-.

Theorem (Ng-Ozsváth-T.)

The pretzel knots $P(-4,-3,3)$ and $P(-6,-3,3)$ have two trans. reps. with same sl, one with $\delta_1 \circ \widehat{\theta} = 0$ and one with $\delta_1 \circ \widehat{\theta} \neq 0$.

We can also use the action of the mapping class group.

Theorem (Ng-Ozsváth-T., using Naturality)

The twist knot 7_2 has two transverse representatives with the same sl, with $\widehat{\theta}$ in different orbits of the mapping class group.

But θ is not a complete invariant: Birman and Menasco have classified closed 3-braids up to transverse isotopy.
In their small examples of non-simple transverse knots, θ lives in a 1-dimensional space, so cannot distinguish the two.