

Heegaard Floer Homology  
Lecture 1: HF Homology from Grid Diagrams

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<http://www.math.columbia.edu/~dpt/speaking>

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# Outline

## ► Heegaard Floer homology in context

Grid diagrams

Computing  $\widehat{HFK}$

General structure of HF homology

# Many equivalent theories

Several theories giving 4-manifold invariants:

Donaldson theory

$\leftrightarrow$  (conj. Seiberg–Witten '94)

Monopoles (Seiberg–Witten)

$\cong$  (Taubes '08)

Embedded contact homology (ECH)

$\cong$  (Cutluhan-Lee-Taubes, Colin-Ghiggini-Honda '10)

Heegaard Floer (HF) homology

At least for 3-mfld  
✓ invariants

We topologists only have one trick!

Each theory has advantages.

Focus

$\widehat{HF}$  homology, the most computable.

## TQFT-like structure

Geometry	Algebra
Closed 4-manifold $W^4$ , $\text{Spin}^c$ structure $\mathfrak{s}$	Invariant $HF(W, \mathfrak{s})$
3-manifold $Y^3$ , $\text{Spin}^c$ structure $\mathfrak{s}$	Homology theory $HF(Y, \mathfrak{s})$
Cobordism $\partial W^4 = (-Y_1) \cup Y_2$	$HF(W) : HF(Y_1) \rightarrow HF(Y_2)$

Subtleties omitted here. E.g., if you make an ordinary TQFT, invariants of closed 4-manifolds are 0.

Similar story for knots in 3-manifolds with surface cobordisms.

### Focus

$HFK(S^3, K; \mathbb{F}_2)$ , the easiest to understand

## Knot homologies in general

Many knot invariants are one- or two-variable Laurent polynomials, associated to quantum groups.

Can often find a doubly- or triply-graded homology theory whose Euler characteristic is the polynomial invariant.

Group	Knot poly	Knot homology
$SL(2)$	Jones $J(t)$	Khovanov (1999)
$SL(n)$	HOMFLY $H(a, z)$	$\left\{ \begin{array}{l} \text{Kh-Roz (2004) } (n \in \mathbb{Z}) \\ \text{Kh-Roz (2005) } (n \text{ variable}) \end{array} \right.$
$GL(1   1)$	Alexander $\Delta(t)$	$\left\{ \begin{array}{l} \text{Heegaard Floer} \\ \text{Seiberg-Witten Floer} \end{array} \right.$
$OSp(n)$	Kauffman $F(a, z)$	Kh-Roz (2007) (conjectural)

*Categorification* is passage polynomial  $\Rightarrow$  homology.

# Properties of $\widehat{HFK}$

$$\dim(\widehat{HFK}_i(K; s)): (\kappa = 10_{132})$$



Characteristics of  $\widehat{HFK}$ :

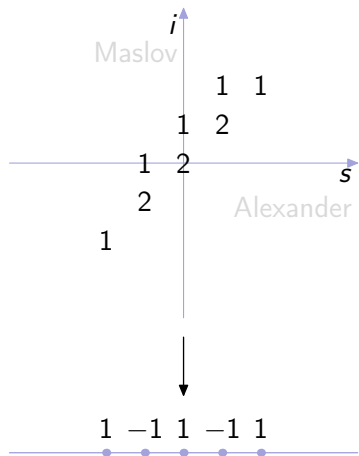
- ▶ **Bigraded;**
- ▶ Euler characteristic is Conway-Alexander polynomial;
- ▶ Max grading is knot genus (so detects unknot); (Ozsváth-Szabó 2001)
- ▶ Determines knot fibration; (Ghiggini, Ni 2006)
- ▶ Defined via pseudo-holomorphic curves.

We will give a simple algorithm for computing  $HFK$ ...

...and so the world's simplest algorithm for knot genus!

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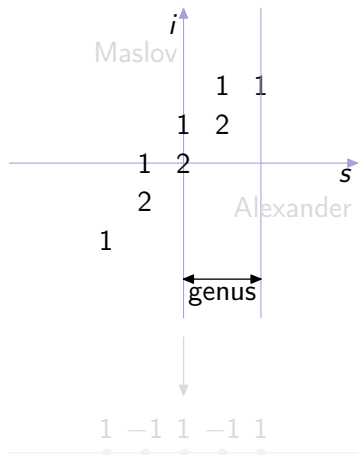
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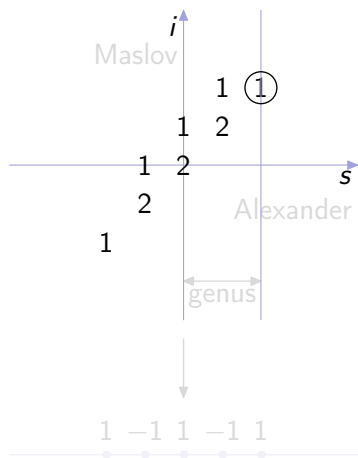
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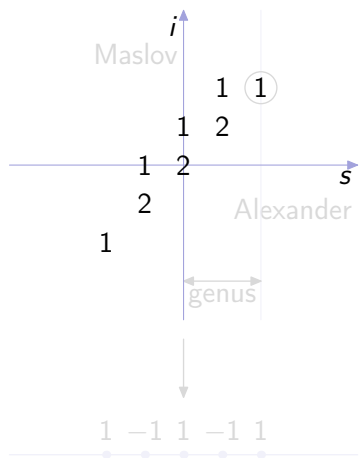
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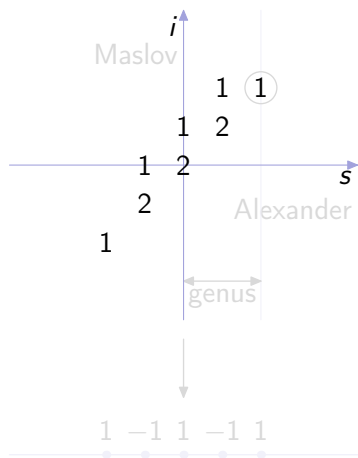
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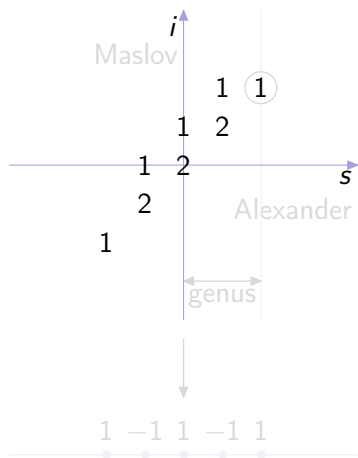
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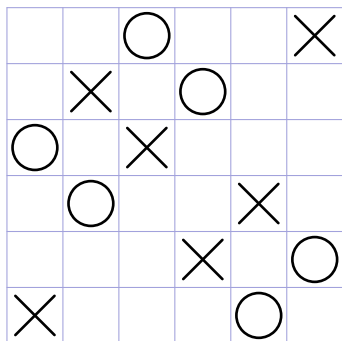
Heegaard Floer homology in context

## ► Grid diagrams

Computing  $\widehat{HFK}$

General structure of HF homology

## Setting: Grid diagrams



Grid diagram: square diagram with one  $X$  and one  $O$  per row and column.

Turn it into a knot: connect  
 $X$  to  $O$  in each column;  
 $O$  to  $X$  in each row.

Cross vertical strands over horizontal.

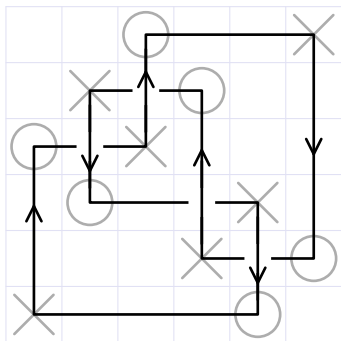
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The knot is unchanged under  
*cyclic rotations*:

Move top segment to bottom.

Think about diagram on a torus.

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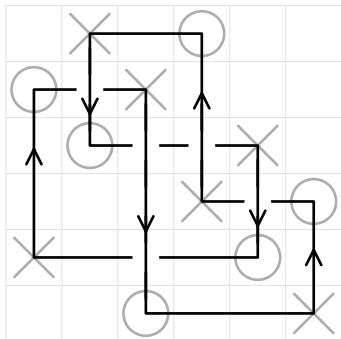
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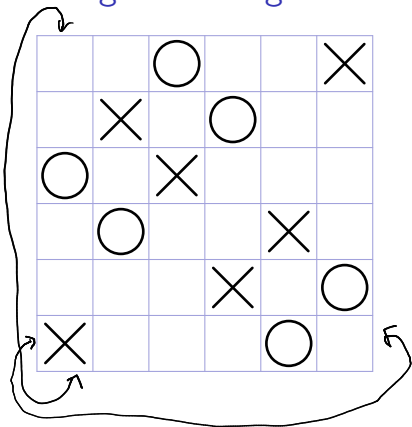
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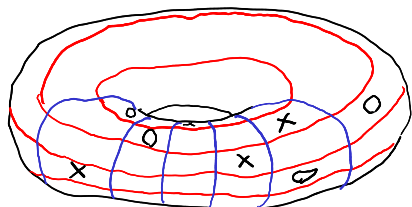
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# Computing the Alexander polynomial

We categorify the following formula:

$$\begin{vmatrix}
 1 & 1 & 1 & t & t & t \\
 1 & 1 & t^{-1} & 1 & t & t \\
 1 & t & 1 & 1 & t & t \\
 1 & t & t & t & t^2 & t \\
 1 & t & t & t & t & 1 \\
 1 & 1 & 1 & 1 & 1 & 1
 \end{vmatrix}
 = \pm t^*(1-t)^{n-1} \Delta(K; t)$$

- ▶ Make matrix of  $t^{-\text{winding \#}}$   
(with extra row/column of 1's);
- ▶ det determines the Conway-Alexander polynomial  $\Delta$   
( $n$  = size of diagram; here 6)

# Outline

Heegaard Floer homology in context

Grid diagrams

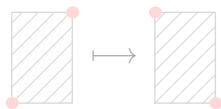
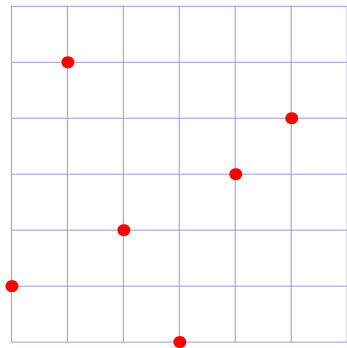
► Computing  $\widehat{HFK}$

General structure of HF homology

# Computing $\widehat{HFK}$ : Chain complex $\widetilde{CK}$

Define a chain complex  $\widetilde{CK}$  over  $\mathbb{F}_2$ .

- ▶  $n!$  generators: matchings between horizontal and vertical gridcircles (as counted in  $\det$  for Alexander).
- ▶ Boundary  $\partial$  switches corners on *empty rectangles*:

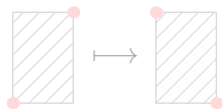
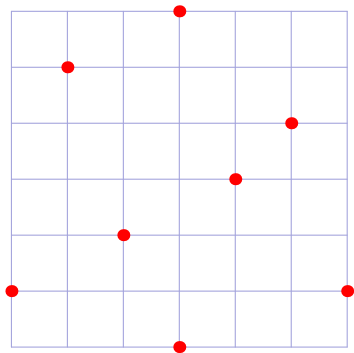


Sum over all ways to switch SW-NE corners of an empty rectangle to NW-SE corners. (*Empty* means: no  $X$ 's,  $O$ 's, or other points in generator.)

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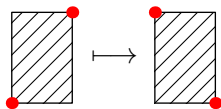
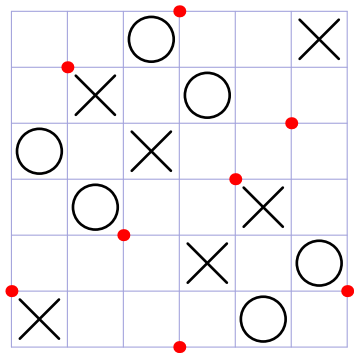


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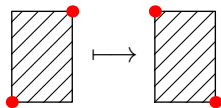
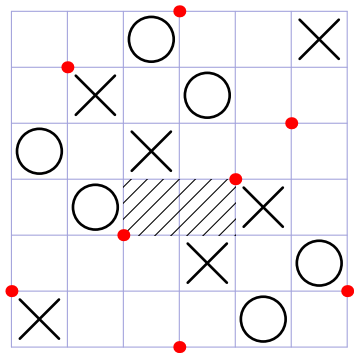


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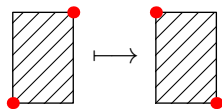
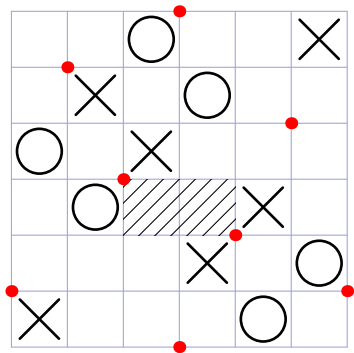


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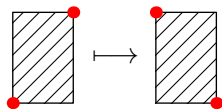
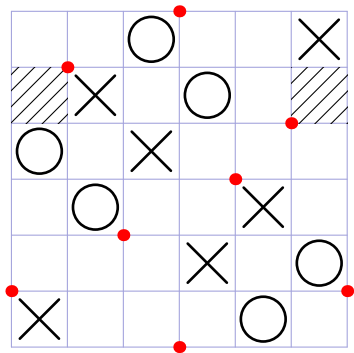


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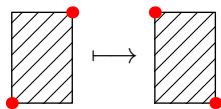
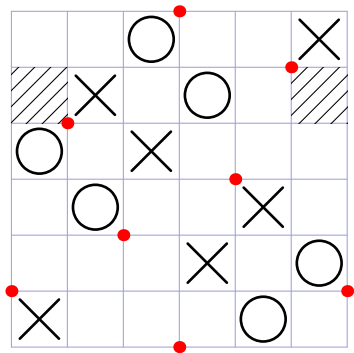


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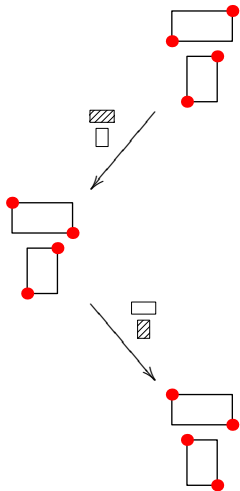
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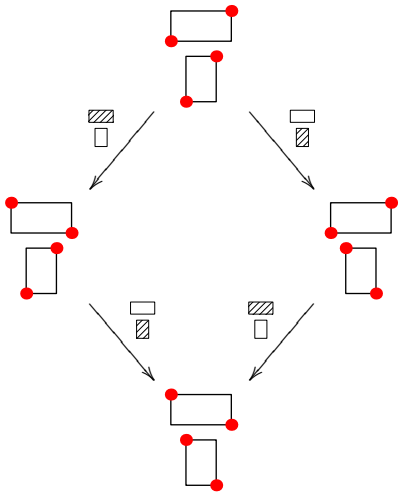
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- ▶ If rectangles are disjoint, take rectangles in either order.
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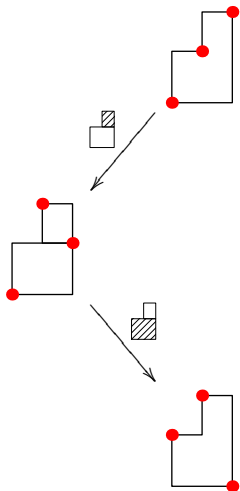
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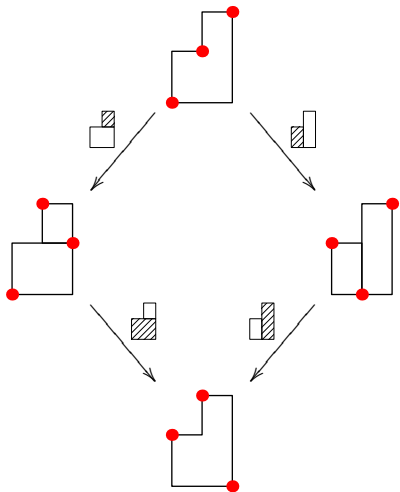
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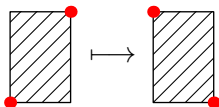


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## Computing $\widehat{HFK}$ : Gradings on $\widetilde{CK}$

In the plane,



removes one *inversion*.

For  $A, B, C \subset \mathbb{R}^2$ ,

$$\begin{aligned} \mathcal{I}(A, B) &:= \#\{ a \square b \mid a \in A, b \in B \} \\ \mathcal{I}(A - B, C) &:= \mathcal{I}(A, C) - \mathcal{I}(B, C) \end{aligned}$$

For  $\mathbf{x}$  a generator,  $\mathbb{X}$  = set of  $X$ 's,  $\mathbb{O}$  = set of  $O$ 's, gradings are:

- ▶ **Maslov:**  $M(\mathbf{x}) := \mathcal{I}(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) + 1$ .
- ▶ **Alexander:** Sum of winding numbers around generator pts, or  $A(\mathbf{x}) := \frac{1}{2}(\mathcal{I}(\mathbf{x} - \mathbb{O}, \mathbf{x} - \mathbb{O}) - \mathcal{I}(\mathbf{x} - \mathbb{X}, \mathbf{x} - \mathbb{X}) - (n - 1))$ .

## Computing $\widehat{HFK}$ : The answer

### Theorem (Manolescu-Ozsváth-Sarkar)

For  $G$  a grid diagram for  $K$ ,

$$H_*(\widetilde{CK}(G)) \simeq \widehat{HFK}(K) \otimes V^{\otimes n-1}$$

where  $V := (\mathbb{F}_2)_{0,0} \oplus (\mathbb{F}_2)_{-1,-1}$ .

(Remember the factor of  $(1-t)^{n-1}$  in determinant formula for  $\Delta$ .)

Gillam and Baldwin used this to compute  $\widehat{HFK}$  for all knots with  $\leq 11$  crossings, including new values of knot genus.

# Outline

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Grid diagrams

Computing  $\widehat{HFK}$

► **General structure of HF homology**

## Heegaard diagrams

Heegaard diagram  $\mathcal{H}$ : surface  $\Sigma$  with two sets of marked curves

$$\alpha = \bigcup \alpha_i, \beta = \beta_i$$

(No intersection within  $\alpha, \beta$ )



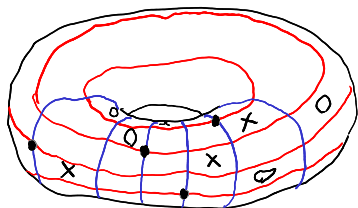
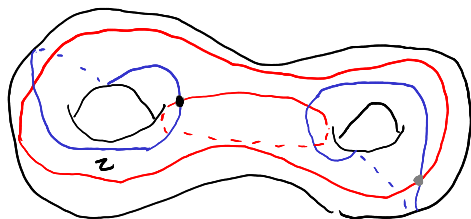
Represents a 3-manifold:

- ▶ Take  $\Sigma \times [0, 1]$
- ▶ Attach handles on  $\alpha \times \{0\}, \beta \times \{1\}$
- ▶ Cap off boundaries

## Complex $CF(\mathcal{H})$

Generators: collections of points in  $\alpha \cap \beta$  with

- ▶ One point on each  $\alpha_i$
- ▶ One point on each  $\beta_i$

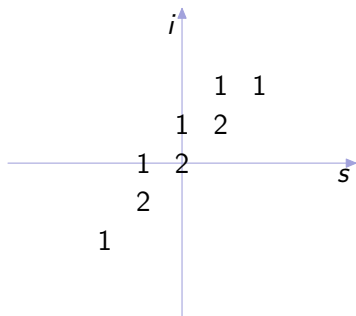


Differential: Count pseudo-holomorphic curves

- ▶ In  $\text{Sym}^k(\Sigma)$  (Ozsváth-Szabó)
- ▶ In  $\Sigma \times [0, 1] \times \mathbb{R}$  (Cylindrical, Lipshitz)

## Another variant: $HF^-$

$\dim \widehat{HFK}_i(K; s):$



To remove factors of  $V^{\otimes n-1}$ :

Complex  $HFK^-$ : variant of  $\widehat{HFK}$

Module over  $\mathbb{F}_2[U]$

$U$  has degree  $(-1, -2)$

Related to  $\widehat{HFK}$  by Universal Coefficient  
Theorem (set  $U$  to 0 on chains).

To compute: Add one  $U_i$  for each  $O$ .

Complex  $CK^-(G)$  over  $\mathbb{F}_2[U_1, \dots, U_n]$

$\partial$  counts rects. that contain only  $O$ 's,  
weighted by corresponding  $U_i$ .

### Theorem

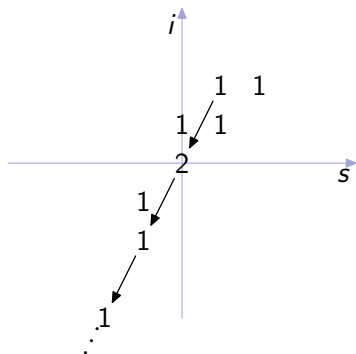
(Manolescu-Ozsváth-Sarkar)

$H_*(CK^-(G)) \simeq HFK^-(K)$ .

Each  $U_i$  acts by  $U$  on the homology.

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Each  $U_i$  acts by  $U$  on the homology.

## Another variant: $HF^-$

To remove factors of  $V^{\otimes n-1}$ :

Complex  $HFK^-$ : variant of  $\widehat{HFK}$

Module over  $\mathbb{F}_2[U]$

$U$  has degree  $(-1, -2)$

Related to  $\widehat{HFK}$  by Universal Coefficient  
Theorem (set  $U$  to 0 on chains).

To compute: Add one  $U_i$  for each  $O$ .

Complex  $CK^-(G)$  over  $\mathbb{F}_2[U_1, \dots, U_n]$

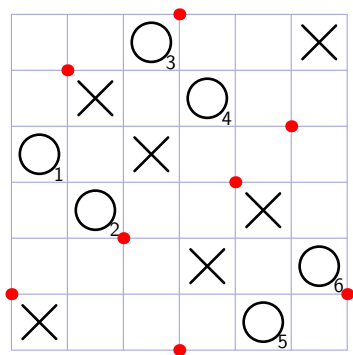
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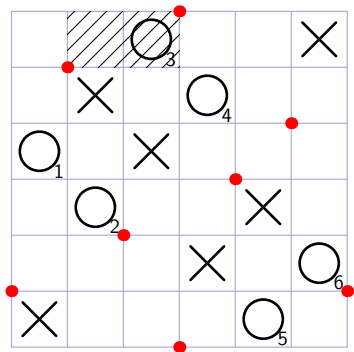
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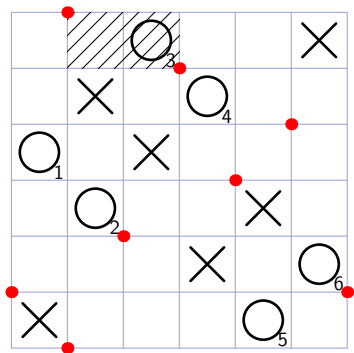
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## Flavors of $HF$

In general, count curves  $S$  with a coefficient of  $U^{n_z(S)}$

$n_z(S) = \#$  of times  $S$  covers basepoint  $z$

Coefficients in	yields
$\mathbb{F}_2[U]/(U=0)$	$\widehat{HF}(Y)$
$\mathbb{F}_2[U]$	$HF^-(Y)$
$\mathbb{F}_2[U, U^{-1}]$	$HF^\infty(Y)$ (determined by $H^*(Y)$ )
$\mathbb{F}_2[U, U^{-1}]/\mathbb{F}_2[U]$	$HF^+(Y)$

Compare: Equivariant cohomology, homology of total space, homology of fixed set

Exact triangle:

$$\dots \rightarrow HF^-(Y) \rightarrow HF^\infty(Y) \rightarrow HF^+(Y) \rightarrow \dots$$

## 4-manifold invariants: The problem

Geometry	Algebra
3-manifold $Y^3$ , Spin <sup>c</sup> structure $\mathfrak{s}$	Homology theory $HF(Y, \mathfrak{s})$
Cobordism $\partial W^4 = (-Y_1) \cup Y_2$	$HF(W) : HF(Y_1) \rightarrow HF(Y_2)$

Works for any of  $\widehat{HF}$ ,  $HF^-$ ,  $HF^\infty$ ,  $HF^+$

Usual way to get invariant of  $W^4$ : cobordism from  $S^3$  to  $S^3$

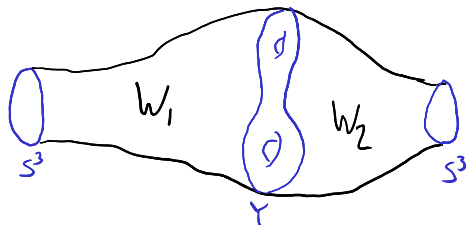
Get zero!

## 4-manifold invariants: The solution

### Lemma

If  $W^4$  is a cobordism with  $b_2^+(W) > 0$ , then  $HF^\infty(W) = 0$  as map.

For  $b_2^+(W) \geq 2$ , split  $W = W_1 \cup_Y W_2$



$$\begin{array}{ccc} & HF^\infty(Y) & \\ & \downarrow & \\ & HF^+(Y) \xrightarrow{HF^+(W_2)} HF^+(S^3) & \\ & \downarrow & \\ HF^-(S^3) \xrightarrow{HF^-(W_1)} & HF^-(Y) & \\ & \downarrow & \\ & HF^\infty(Y) & \end{array}$$

## Appendix: Crossing number vs. Grid number

Knots are usually ordered by *crossing number*:

Minimum number of crossings in a planar diagram.

For grid diagrams, natural to consider *grid number* (or *arc index*):

Minimum size of a grid diagram.

Theorem (Bae–Park, Morton–Beltrami)

*Grid number of an alternating knot is equal to crossing number + 2.*

*For non-alternating knots, grid number strictly less.*