

Algebraic Topology II, Problem Set 9

Due Thursday, April 24, 2008

1. [from Milnor–Stasheff, Problem 9-C] Let $\Delta \subset S^n \times S^n$ be the diagonal and let $A \subset S^n \times S^n$ be the anti-diagonal (pairs of antipodal vectors). Show that the total space E of the TS^n , the tangent bundle to S^n , is homeomorphic to $(S^n \times S^n) \setminus A$. Deduce that

$$H^*(E, E_0) \simeq H^*(S^n \times S^n, S^n \times S^n \setminus \Delta) \simeq H^*(S^n \times S^n, A) \subset H^*(S^n \times S^n).$$

If n is even, show that the Euler class $e(TS^n)$ is twice a generator of $H^n(S^n; \mathbb{Z})$. Deduce that in this case TS^n has no non-trivial subbundles.

2. [Milnor–Stasheff, Problem 11-D] Prove that all Stiefel–Whitney numbers of a 3-manifold are zero.
3. Given an n -dimensional vector bundle η over a base space X . Show that $S\eta$ is a principal fibration over X iff η is oriented.
4. Given a CW-fibration $\pi : X \rightarrow Y$ with fiber F , show that if F is n -connected, then for any other CW-complex W and map $f : W \rightarrow Y$, there is a unique lift (up to homotopy) of f from W^n (the n -skeleton of W) to X . [We used this in class to show that the principal obstruction is canonical.]
5. [from Milnor–Stasheff, Problem 14-C] As in class, let $V_k(\mathbb{C}^n)$ be the space of all complex k -frames in \mathbb{C}^n . Show that $V_k(\mathbb{C}^n)$ is $2(n-k)$ connected, and that

$$\pi_{2(n-k)+1}(V_k(\mathbb{C}^n)) \cong \mathbb{Z}.$$

For ω a complex n -dimensional bundle, show that the principle obstruction to the existence of a cross-section of $V_{n-k}(\omega)$ is the k 'th Chern class. (Start by showing it lies in the correct group.)

6. [from Milnor–Stasheff, Problem 14-B] Show that, if ω is an n -dimensional complex bundle, then $w(\omega_{\mathbb{R}})$ is the reduction of $c(\omega)$ to coefficients modulo 2. In particular the odd Stiefel–Whitney classes of $\omega_{\mathbb{R}}$ vanish.