Calculus 3: Notes on representations of lines in the plane

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There are two natural ways to represent lines in the plane: with an *implicit equation*, like

$$2x + 3y - 2 = 0 \tag{1}$$

(where the line is the set of points (x, y) satisfying the equation above), and *parametrically*, like

$$x(t) = 1 + 2t \qquad y(t) = 0 - t \tag{2}$$

(where the line is the set of points $(x(t), y(t))$ as t varies over all real numbers).

In order to convert between these two representations, we first write both in terms of vectors. Setting

$$\vec{r} = \langle x, y \rangle \qquad \text{(the position vector)} \tag{3}$$

$$\vec{n} = \langle 2, 3 \rangle \qquad \text{(the vector of coefficients)} \tag{4}$$

we see that Equation (1) can be written

$$\vec{n} \cdot \vec{r} = 2. \tag{5}$$

That is, $\vec{n} \cdot \vec{r}$ is a constant, independent of the point $\vec{r}$ on the line. Let $\vec{r_0}$ be some arbitrary fixed point on the line; then we can rewrite (5) as

$$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0} \tag{6}$$

$$\vec{n} \cdot \vec{r} - \vec{n} \cdot \vec{r_0} = 0 \tag{7}$$

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0. \tag{8}$$

That is, the vector $\vec{n}$ is perpendicular to the vector $\vec{r} - \vec{r_0}$. Since $\vec{r}$ and $\vec{r_0}$ are both on the line, $\vec{r} - \vec{r_0}$ is *parallel* to the line (Note the difference!) and so $\vec{n}$ is *perpendicular* to the line; it is called a *normal* vector.

In a parametric equation like (2), keep $\vec{r}$ as before and set

$$\vec{v} = \langle 2, -1 \rangle \qquad \text{(the vector of coefficients of } t\text{)} \tag{9}$$

$$\vec{r_0} = \langle 1, 0 \rangle \qquad \text{(the position at } t = 0\text{)} \tag{10}$$

Equation (2) can be rewritten

$$\vec{r}(t) = \vec{r_0} + t\vec{v}. \tag{11}$$

If we evaluate at time $t = 1$ and subtract, we see that

$$\vec{r}(1) - \vec{r}(0) = (\vec{r0} + \vec{v}) - \vec{r_0}) = \vec{v} \tag{12}$$

so $\vec{v}$ is the difference between two vectors on the line, so is *parallel* to the line.

Thus an implicit equation for a line in the plane is related to a vector $\vec{n}$ perpendicular to the line, while a parametric equation is related to a vector $\vec{v}$ parallel to the line. To convert from one to the other, we need to switch the two; that is, rotate a vector 90 degrees. Note that this is only well-defined (up to negating the vector) because we are in the plane—in space there are many vectors perpendicular to a given one.

Let's look at an example: let's convert Equation (1) to parametric form. We already wrote it in vector form in (5). To write it in vector form, we first find a vector $\vec{v}$ parallel to the line:

$$\vec{n} = \langle 2, 3\rangle \tag{13}$$

$$\vec{v} = \vec{n}_\perp = \langle 3, -2\rangle \tag{14}$$

Now we find a point $\vec{r_0}$ on the line. We can do this by finding the x-intercept; that is, simultaneously solving (1) and $y = 0$:

$$2x + 3y - 2 = 0 \tag{15}$$

$$x = 0 \tag{16}$$

$$3y = 2 \tag{17}$$

$$y = 2/3 \tag{18}$$

$$\vec{r_0} = \langle 0, 2/3\rangle \tag{19}$$

Then the parametric equation is

$$\vec{r}(t) = \vec{r_0} + t\vec{v} \tag{20}$$

or, in non-vector form,

$$x(t) = 3t \qquad y(t) = 2/3 - 2t. \tag{21}$$

In 3 dimensions, equations for planes are usually given in implicit form, with a vector perpendicular to the plane, while equations for lines are usually given in parametric form, with a vector parallel to the line.

Exercises

- Verify the conversion from parametric to implicit by plugging (21) into (1) and checking that you get 0, regardless of what t is.
- Convert Equation (2) from parametric to implicit form.