MONOPOLES AND FUETER SECTIONS ON THREE-MANIFOLDS

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Well-versed in the expanses
that stretch from earth to stars,
we get lost in the space
from earth up to our skull.

Intergalactic reaches
divide sorrow from tears.
En route from false to true
you wither and grow dull.

— Wisława Szymborska, *To My Friends*[1]

Dedicated to my parents and grandparents,
who taught me to navigate 'the space from earth up to my skull'

1 *Poems New and Collected*, transl. Stanisław Barańczak and Clare Cavanagh, Harcourt Inc., 1998.

ABSTRACT

The subject of this dissertation is a relationship between two classes of solutions to elliptic differential equations on three-manifolds: monopoles and Fueter sections. The main result is a description of a wall-crossing phenomenon for a signed count of monopoles. As a corollary, we prove the existence of Fueter sections with singularities. We also study monopoles and Fueter sections using methods of complex geometry, on three-manifolds which are the product of a circle and a surface. Finally, we discuss the relevance of our results to gauge theory on higher-dimensional Riemannian manifolds with special holonomy.

DECLARATION

This dissertation is based on the articles [Doa17a; Doa17b; DW17a; DW17b; DW17c], the last three of which were written in collaboration with Thomas Walpuski. In Section 1.6 and at the beginning of each chapter, we explain which parts were written in collaboration, and refer to the relevant sections of the articles.

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CONTENTS

1 INTRODUCTION

1.1 GAUGE THEORY AND INVARIANTS OF MANIFOLDS

Over the last thirty years, the study of the equations of gauge theory has led to spectacular advances in low-dimensional topology. These developments trace back to Donaldson's construction of invariants of smooth 4–manifolds using instantons—a special class of solutions to the Yang–Mills equations [Don83; Don87; Don90]. Floer extended Donaldson's ideas to 4–manifolds with boundary and introduced invariants of 3–manifolds, which behave functorially with respect to cobordisms, i.e. define a $3+1$–dimensional topological field theory [Flo88; Ati88a; Ati88b]. Surfaces can be incorporated in this theory as well, by considering flat connections in dimension two.

dimension	invariant	type of invariant
4	Donaldson invariants	number
3	Instanton Floer homology	vector space
2	Fukaya category of moduli space	category

The basic ideas behind Donaldson–Floer theory are:

1. The Donaldson invariants of a closed 4–manifold X are defined by integrating certain cohomology classes over the moduli space of instantons on X. In the simplest situation when the moduli space is zero-dimensional, the invariant is a signed count of instantons.

2. The instanton Floer homology groups of a closed 3–manifold Y are defined by formally applying the construction of Morse homology to the Chern–Simons functional—a functional on the infinite-dimensional space of connections on Y, whose critical points are flat connections on Y, and whose gradient flowlines correspond to instantons on the cylinder $X = Y \times \mathbf{R}$.

3. For a closed surface Z, the moduli space $\mathcal{M}_Z$ of irreducible flat connections on Z is naturally a symplectic manifold [AB83]. For every symplectic manifold there is an associated Fukaya category, whose objects are Lagrangian submanifolds and morphisms spaces are constructed using pseudo-holomorphic discs with boundary on such submanifolds; see, for example, [Aur14; Smi15]. The Atiyah–Floer conjecture and its various refinements relate the algebraic structures of the Fukaya category of $\mathcal{M}_Z$ to the instanton invariants of 3–manifolds admitting a Heegaard splitting along Z [Ati88a; Sal95; DF18]. The conjecture is based on the observation that instantons on $X = Z \times \mathbf{R}^2$ give rise to pseudo-holomorphic discs in $\mathcal{M}_Z$.

There is a parallel topological field theory based on the Seiberg–Witten equation on 4–manifolds and its dimensional reductions to 3–manifolds and surfaces [Mor96; KM07; OS04]. It is natural to ask whether equations of gauge theory other than the instanton and Seiberg–Witten equations can be used to construct invariants of manifolds. The subject of this dissertation fits into two new, quickly developing lines of research whose goal is to extend the scope of applications of gauge theory to geometry:

1. *Beyond Yang–Mills & Seiberg–Witten.* While most of research in mathematical gauge theory so far focused on the Yang–Mills and Seiberg–Witten equations, physicists have long considered other interesting differential equations on low-dimensional manifolds. These include the Kapustin–Witten equation, which is expected to lead to new insights into the topology of 3–manifolds and knots [KW07; Wit18], and the Vafa–Witten equation, which has links to algebraic geometry [VW94]. There has been recently a surge in the study of these and related equations, initiated by deep work of Taubes [Tau13b; Tau13a; Tau14].

2. *Beyond low dimensions.* Many attractive features of Yang–Mills theory in low dimensions generalize to manifolds of dimensions six, seven, and eight, provided that they are equipped with a Riemannian metric of special holonomy, given by the Lie groups $\mathrm{SU}(3)$, G_2, and $\mathrm{Spin}(7)$, respectively. Donaldson, Thomas, and E. Segal [DT98; DS11] proposed to define invariants of special holonomy manifolds using Yang–Mills theory, mimicking Donaldson–Floer theory in low dimensions. Such invariants would be helpful in classifying the existing millions of examples of special holonomy manifolds.

There is a rather surprising connection between these two topics, which was discovered by Walpuski [Wal13; Wal17], building on ideas of Donaldson–Segal [DS11] and Haydys [Hay12]. In the next three sections, we discuss two interesting classes of solutions to elliptic differential equations on low-dimensional manifolds and explain a conjectural relationship between them. We then relate this discussion to higher-dimensional Yang–Mills theory. At the end of the introduction, we summarize the main results proved in this dissertation.

1.2 MONOPOLES AND QUATERNIONIC REPRESENTATIONS

The discovery of the Seiberg–Witten equation gave mathematicians powerful tools for studying low-dimensional manifolds. While originally Seiberg–Witten theory concerned 4–manifolds, in this dissertation we focus on the simpler, 3–dimensional version of the equation. Over a Riemannian 3–manifold M, equipped with a spin structure and a $\mathrm{U}(1)$–bundle, the Seiberg–Witten equation is

$$\begin{cases} \not{D}_A \Phi = 0, \\ F_A = \mu(\Phi). \end{cases} \tag{1.2.1}$$

A solution, or a *monopole*, is a pair (Φ, A) consisting of a section Φ of the spinor bundle, and a $\mathrm{U}(1)$–connection A. Here, $\not{D}_A$ denotes a twisted Dirac operator, F_A is the curvature of A, and $\mu(\Phi)$ is a certain quadratic function of Φ. One considers two solutions equivalent if they differ by a $\mathrm{U}(1)$–gauge transformation.

It turns out that there is a broad class of partial differential equations whose general form is the same as that of (1.2.1), and similarly for the 4–dimensional version of the equation; see [Tau99; Pid04a; Hay08] and [Nak15, Section 6]. This theory will be reviewed in Section 2.2. The main idea is that there exists such an equation for every choice of a compact Lie group G together with a quaternionic representation $\rho\colon G \to \mathrm{Sp}(S)$. Here, $\mathrm{Sp}(S)$ denotes the group of quaternion-linear isometries of a quaternionic vector space S. We call the resulting equation the *Seiberg–Witten equation associated with* ρ and its solutions *ρ–monopoles*. A ρ–monopole consists of a section Φ of a vector bundle on M whose fiber is S, and a connection A on a fixed principal G–bundle on M. The pair (Φ, A) satisfies equation (1.2.1), where now μ is obtained from the hyperkähler moment map—a natural quadratic map associated with the representation ρ. One considers two such pairs equivalent if they differ by a G–gauge transformation.

We recover the classical Seiberg–Witten equation by setting $G = \mathrm{U}(1)$, with ρ being the standard representation on the space of quaternions $\mathbf{H} = \mathbf{C}^2$. In fact, almost every equation studied so far in mathematical gauge theory arises from this construction or its 4–dimensional version.

Regarding all the applications of Seiberg–Witten theory to low-dimensional topology, it is natural to wonder whether other Seiberg–Witten equations lead to topological invariants. The expected dimension of the moduli space of ρ–monopoles on a 3–manifold is zero, so one hopes to define a signed count of solutions on any such manifold M, equipped with a spin structure and a principal G–bundle. Ignoring various technical difficulties involved in defining such a signed count, we expect to obtain an integer

$$n(M, \mathbf{p}) = \sum_{\rho\text{–monopoles } [\Phi, A]} \mathrm{sign}(\Phi, A), \tag{1.2.2}$$

where the sum is taken over all gauge equivalence classes of ρ–monopoles and $\mathrm{sign}(\Phi, A) = \pm 1$ is defined using an orientation procedure that will be explained later. Here $\mathbf{p}$ denotes all continuous parameters of the equation (1.2.1), such as the choice of a Riemannian metric on M, or its perturbations, for example holonomy perturbations or abstract perturbations of the equation thought of as a Fredholm map between Banach manifolds, cf. [DK90, Section 4.3.6], [KM07, Section 10]. We are deliberately vague here about what $\mathbf{p}$ exactly is, as the details will depend on ρ and M. The point is that we need to allow a sufficiently broad class of parameters, $\mathscr{P}$ say, so that for a generic $\mathbf{p} \in \mathscr{P}$ the moduli space of ρ–monopoles is zero-dimensional, i.e. consists of isolated points. If the number of points is finite, we can define $n(M, \mathbf{p})$ and ask how it depends on the choice of $\mathbf{p} \in \mathscr{P}$.

Example 1.2.1. For $G = \mathrm{SU}(2)$ and $S = \{0\}$, ρ–monopoles are simply flat $\mathrm{SU}(2)$–connections. Taubes [Tau90] proved that, if M is a homology sphere, a signed count of suitably perturbed flat $\mathrm{SU}(2)$–connections is the Casson invariant of M. A similar result is true when $b_1(M) \geqslant 1$, except in this case it is more convenient to consider flat $\mathrm{SO}(3)$–connections on a non-trivial $\mathrm{SO}(3)$–bundle in order to avoid reducible solutions [Don02, Section 5.6], [Pou15].

Example 1.2.2. For the classical Seiberg–Witten equation, i.e. $G = \mathrm{U}(1)$ and $S = \mathbf{H}$, Meng and Taubes [MT96] showed that the number $n(M, \mathbf{p})$ does not depend on the choice of a generic $\mathbf{p}$, provided that $b_1(M) > 1$. In fact, $n(M, \mathbf{p})$ is equal to the sum of the coefficients of the Alexander polynomial of M, a well-known topological invariant. The condition on b_1 is necessary to exclude reducible solutions, which spoil the invariance of $n(M, \mathbf{p})$ as $\mathbf{p}$ varies.

Reducible solutions are likely to play an important role in the study of the Seiberg–Witten equations associated with other quaternionic representations. It is possible that one cannot define sensibly the number $n(M, \mathbf{p})$ for a general 3–manifold, as it is the case for the classical Seiberg–Witten equation, and the right approach is to develop a more complicated theory involving reducibles, such as equivariant Floer theories constructed using instantons [Flo88; Don02] or classical Seiberg–Witten monopoles [KM07]. Details will depend, of course, on the choice of G and ρ.

However, we will see that even when we ignore the issues related to reducibles, we should not expect $n(M, \mathbf{p})$ to be independent of $\mathbf{p}$ for more complicated quaternionic representations. The reason is that, unlike in the two classical cases described above, the moduli space of ρ–monopoles might be non-compact for some choices of $\mathbf{p}$. This is a completely new phenomenon, which was discovered by Taubes [Tau13b; Tau13a] and studied further in [HW15; Tau16; Tau17]. In order to understand it, we need to consider another elliptic differential equation.

1.3 FUETER SECTIONS

Spinors in dimensions three and four are intimately related to the algebra of quaternions $\mathbf{H}$, which can be interpreted as the spinor space of $\mathbf{R}^3$. Under this identification, the Dirac equation for a spinor on $\mathbf{R}^3$, i.e. a map $s\colon \mathbf{R}^3 \to \mathbf{H}$, is

$$i\frac{\partial s}{\partial x} + j\frac{\partial s}{\partial y} + k\frac{\partial s}{\partial z} = 0. \tag{1.3.1}$$

The quaternionic viewpoint allows us to generalize the Dirac equation to a non-linear elliptic differential equation for sections of certain fiber bundles over 3–manifolds. (There is also a corresponding 4–dimensional equation.) We now give an overview of this theory, referring to [Hay08; Hay12; Hay14a; HNS09; Sal13; Wal15] and Section 2.5 for more details.

Let M be an oriented Riemannian 3–manifold, and let $\pi\colon \mathfrak{X} \to M$ be a fiber bundle whose fibers are hyperkähler manifolds. Suppose that we are given an identification of the SO(3)–frame bundle of M with the SO(3)–bundle whose fiber at a point $x \in M$ consists of all hyperkähler triples on $\pi^{-1}(x)$, i.e. triples (I, J, K) of orthogonal complex structures satisfying the quaternionic relations $IJ = K$, etc. In particular, every unit vector in T_xM—or, using the Riemannian metric, every unit covector in T^*_xM—gives rise to a complex structure on $\pi^{-1}(x)$. This induces a bundle homomorphism

$$\gamma\colon\ \pi^*TM \otimes V\mathfrak{X} \to V\mathfrak{X},$$

where $V\mathfrak{X} = \ker \pi_*$ is the vertical subbundle of the tangent bundle of $\mathfrak{X}$. The map γ is a generalization of the Clifford multiplication for the spinor bundle. Suppose also that we have a connection on $\pi\colon \mathfrak{X} \to M$ which preserves γ.

Given this data, one defines the *Fueter operator*, a first order non-linear elliptic operator acting on sections of $\mathfrak{X}$. For a section $s\colon\ M \to \mathfrak{X}$, the projection of the differential $\mathrm{d}s\colon TM \to s^*T\mathfrak{X}$ on the vertical subbundle $V\mathfrak{X} \subset T\mathfrak{X}$ gives us a section ∇s of $T^*M \otimes s^*V\mathfrak{X}$. The Fueter operator is defined by

$$\mathfrak{F}(s) = \gamma(\nabla s).$$

Locally, if we choose an orthonormal frame of TM and denote by (I, J, K) the corresponding family of hyperkähler triples on the fiber of $\mathfrak{X}$, the Fueter operator is given by a formula generalizing (1.3.1):

$$\mathfrak{F}(s) = I(s)\nabla_1 s + J(s)\nabla_2 s + K(s)\nabla_3 s.$$

A *Fueter section* is a solution to the equation $\mathfrak{F}(s) = 0$. This is a non-linear analogue of the Dirac equation. Indeed, the spinor bundle is an example of $\mathfrak{X}$ as above, with the corresponding Fueter operator being the Dirac operator.

The relationship between Fueter sections and ρ–monopoles is based on the *hyperkähler quotient construction* [Hit+87]. Recall that associated with every quaternionic representation $\rho\colon G \to \mathrm{Sp}(S)$ there is a moment map μ and the hyperkähler quotient of ρ is, by definition, $\overline{X} = \mu^{-1}(0)/G$. In general, $\overline{X}$ is a singular variety stratified by hyperkähler manifolds; denote by $X \subset \overline{X}$ the top stratum.

A section Φ appearing in the Seiberg–Witten equation (1.2.1) associated with ρ takes values in a vector bundle whose fiber is the quaternionic space S. By perfoming the hyperkähler quotient construction fiber by fiber, we obtain a bundle $\pi\colon \overline{\mathfrak{X}} \to M$ with fiber $\overline{X}$. Similarly there is a subbundle $\mathfrak{X} \to M$ whose fiber is the hyperkähler manifold X. The geometric data required to formulate the Seiberg–Witten equation equip $\mathfrak{X}$ with the Clifford multiplication γ and a compatible connection. Thus, there is a corresponding Fueter operator $\mathfrak{F}$ acting on sections of $\mathfrak{X}$. If, as before, we denote by $\mathbf{p}$ all continuous parameters of the Seiberg–Witten equation (a Riemannian metric, perturbations of the equation, etc.), then $\mathfrak{F} = \mathfrak{F}_{\mathbf{p}}$ depends on $\mathbf{p}$. As a result, solutions to the Fueter equation $\mathfrak{F}_{\mathbf{p}}(s) = 0$ may or may not exist depending on the choice of $\mathbf{p}$.

The Fueter operator enters the analysis of the Seiberg–Witten equation in the following way. A basic step in defining the number $n(M, \mathbf{p})$ as in (1.2.2) is proving compactness of the moduli space of ρ–monopoles. For the classical Seiberg–Witten equation (i.e. $G = \mathrm{U}(1)$, $S = \mathbf{H}$), this is done by establishing an a priori bound on the L^2–norm of the spinor Φ. For other choices of the quaternionic representation, there is no such a priori bound. However, one can still prove that a sequence of solutions (Φ_i, A_i) has a subsequence convergent modulo gauge, provided that $\|\Phi_i\|_{L^2}$ is bounded. Hence, any non-compactness phenomenon must be related to $\|\Phi_i\|_{L^2}$ going to infinity. With this observation in mind, it is natural to blow-up the Seiberg–Witten equation by setting

$$\widehat{\Phi} = \Phi/\|\Phi\|_{L^2} \quad \text{and} \quad \varepsilon = \|\Phi\|_{L^2}^{-1}.$$

Equation (1.2.1) for (Φ, A) is then equivalent to the following equation for a triple $(\varepsilon, \widehat{\Phi}, A)$ with $\varepsilon > 0$:

$$\begin{cases} \not{D}_A \widehat{\Phi} = 0, \\ \varepsilon^2 F_A = \mu(\widehat{\Phi}), \\ \|\widehat{\Phi}\|_{L^2} = 1. \end{cases} \tag{1.3.2}$$

We are left with the task of analyzing sequences of solutions $(\varepsilon, \widehat{\Phi}, A)$ of equation (1.3.2), with $\varepsilon \to 0$. While such analysis turns out to be quite delicate (more about this in the next section), by formally setting $\varepsilon = 0$ in (1.3.2), we obtain the following equation for a pair $(\widehat{\Phi}, A)$:

$$\begin{cases} \not{D}_A \widehat{\Phi} = 0, \\ \mu(\widehat{\Phi}) = 0, \\ \|\widehat{\Phi}\|_{L^2} = 1. \end{cases} \tag{1.3.3}$$

We therefore expect that the possible non-compactness of the moduli space of ρ–monopoles is caused by solutions $(\widehat{\Phi}, A)$ to (1.3.3). This equation is degenerate and not elliptic, even modulo gauge, because it contains no differential equation on the connection A. However, Haydys proved that there is a natural correspondence between gauge equivalence classes of solutions to (1.3.3) and Fueter sections of $\mathfrak{X}$ [Hay12, Theorem 4.6]

While the full statement of Haydys' theorem, which we present in Section 2.5, is somewhat involved, here is the basic idea. Recall that $\overline{X} = \mu^{-1}(0)/G$ is the fiber of $\overline{\mathfrak{X}} \to M$. Thus, every solution $(\widehat{\Phi}, A)$ of (1.3.3) gives rise to a continuous section s of $\overline{\mathfrak{X}}$. In fact, two gauge equivalent solutions induce the same section of $\overline{\mathfrak{X}}$. Moreover, if the resulting section s takes values in the top stratum $\mathfrak{X} \subset \overline{\mathfrak{X}}$, then the Dirac equation obeyed by $\widehat{\Phi}$ implies that s satisfies the Fueter equation. Finally, Haydys proved that the connection A is, up to gauge, pulled back by s from a certain natural G–connection on $\mathfrak{X}$. Thus, the correspondence $(\widehat{\Phi}, A) \mapsto s$ is a bijection between gauge equivalence classes of solutions to (1.3.3) (for which s takes values in $\mathfrak{X}$) and Fueter sections of $\mathfrak{X}$.

The summary of this discussion is that we should expect the moduli space of ρ–monopoles to be non-compact precisely for those choices of $\mathbf{p}$ for which there exists a Fueter section of the bundle of hyperkähler quotients $\mathfrak{X}$.

1.4 A conjectural picture

We now discuss some open problems and conjectures about ρ–monopoles and Fueter sections. While it might be overly optimistic to expect these conjectures to hold, as stated, for all quaternionic representations and 3–manifolds, it is reasonable to hope that they are true for a broad class of interesting examples.

Fix a closed, oriented 3–manifold M and a quaternionic representation ρ. We consider the associated Seiberg–Witten equation under the following assumptions.

1. *Transversality.* We assume that there is a sufficiently large class $\mathscr{P}$ of parameters, or perturbations, of the Seiberg–Witten equation (1.2.1), so that for a generic choice of $\mathbf{p} \in \mathscr{P}$ the corresponding moduli space of irreducible solutions is zero-dimensional, and for a generic path in $\mathscr{P}$, the corresponding one-parameter moduli space is a one-dimensional manifold with boundary. The perturbations should be chosen in such a way that the compactness statements stated below still hold for the perturbed equation. The space $\mathscr{P}$ should contain at least the space of all Riemannian metrics on M, but it can include also various additional terms added to (1.2.1), or abstract perturbations. The details will depend on the equation. For example, in Chapter 3 we define $\mathbf{p} \in \mathscr{P}$ to be a triple consisting of a Riemannian metric and a closed 2–form on M, and a connection on a certain bundle over M.

2. *Reducibles.* We assume that M and ρ are chosen in such a way that all solutions to the Seiberg–Witten equation are irreducible, i.e. have trivial stabilizer in the gauge group, at least for all $\mathbf{p}$ outside a codimension two subset of $\mathscr{P}$.

3. *Orientations.* There is a standard way of orienting moduli spaces, discussed in Section 2.4, using determinant line bundles. As we will see, given a quaternionic representation ρ, there is a simple criterion to determine whether the determinant line bundle for the corresponding Seiberg–Witten equation is orientable. If this is the case, we equip all the moduli spaces with an orientation, otherwise we will count solutions mod 2.

While (1) is a reasonable assumption, condition (2) will be violated in many cases. As we have mentioned earlier, already for the classical Seiberg–Witten equation over homology 3–spheres or for flat SU(2) connections over 3–manifolds with $b_1 > 0$, reducibles are known to cause serious difficulties. However, in this dissertation we focus on the problem of compactness, which is independent of reducibles, and so it is convenient to assume that condition (2) is satisfied. In Chapter 3, we will consider an interesting example of a representation ρ, for which (2) holds whenever $b_1(M) > 1$, as for the classical Seiberg–Witten equation.

The first conjecture states in a precise way what it means for a sequence of ρ–monopoles to converge to a Fueter section.

Conjecture 1.4.1 (Compactness)**.** *Let (Φ_i, A_i) be a sequence of ρ–monopoles with respect to a sequence of parameters $\mathbf{p}_i$ converging to $\mathbf{p}$. Suppose that*

$$\lim_{i\to\infty} \|\Phi_i\|_{L^2} = \infty$$

and set $\widehat{\Phi}_i = \Phi / \|\Phi_i\|_{L^2}$.

There exist a closed, 1–rectifiable subset $Z \subset M$ and a pair $(\widehat{\Phi}, A)$ defined over $M\backslash Z$ such that the following hold:

1. *After passing to a subsequence and applying gauge transformations, $(\widehat{\Phi}_i, A_i)$ converges to $(\widehat{\Phi}, A)$ in C^∞_{loc} over $M\backslash Z$.*

2. *$(\widehat{\Phi}, A)$ satisfies the limiting equation (1.3.3) with respect to $\mathbf{p}$ over $M\backslash Z$.*

3. *If s denotes a continuous section of $\overline{\mathfrak{X}}$ over $M\backslash Z$ induced from $\widehat{\Phi}$, then s extends to a continuous section of $\overline{\mathfrak{X}}$ over all of M in such a way that*

$$Z = s^{-1}(\overline{\mathfrak{X}}\backslash \mathfrak{X})$$

4. *Over $M\backslash Z$, the section s is smooth and obeys the Fueter equation with respect to $\mathbf{p}$*

$$\mathfrak{F}_{\mathbf{p}}(s) = 0.$$

(Note that the equation makes sense because, by the third condition, s takes values in the smooth fiber bundle $\mathfrak{X} \subset \overline{\mathfrak{X}}$ over $M\backslash Z$.) Moreover, we have

$$\int_{M\backslash Z} |\nabla s|^2 < \infty.$$

To summarize, we expect to be able to pass to the limit $\varepsilon \to 0$ in equation (1.3.2). However, the rescaled sequence $(\widehat{\Phi}_i, A_i)$ converges only outside a certain 1–dimensional set $Z \subset M$, called the *singular set*, along which the limiting Fueter section s takes values in the singular strata of the hyperkähler quotient. A special case is when Z is empty and s is a smooth section of $\mathfrak{X}$ over M.

A version of Conjecture 1.4.1 was proved by Taubes for the Seiberg–Witten equation associated with the adjoint representation

$$G = \mathrm{SU}(2) \quad \text{and} \quad S = \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{su}(2),$$

and for other, closely related equations on 4–manifolds, including the Vafa–Witten and Kapustin–Witten equations [Tau13b; Tau13a; Tau14; Tau16; Tau17]. All later developments on compactness for generalized Seiberg–Witten equations, including the statement of Conjecture 1.4.1, have been inspired by Taubes' work. Haydys and Walpuski [HW15] established a similar compactness result for the Seiberg–Witten equation with multiple spinors, corresponding to the representation

$$G = \mathrm{U}(1) \quad \text{and} \quad S = \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^n.$$

We will discuss their work in detail in Chapter 3. Recent progress by Walpuski and B. Zhang [WZ19] suggests that the currently available techniques might be sufficient to prove Conjecture 1.4.1 for a broad class of representations.

Guided by these compactness results, we extend the definition of a Fueter section so that it includes also *singular Fueter sections*, that is pairs (Z, s) consisting of a 1–rectifiable subset $Z \subset M$ and a continuous section s of $\overline{\mathfrak{X}}$, satisfying conditions (3) and (4) from Conjecture 1.4.1. We will say that a Fueter section is *smooth*, or *non-singular*, if the set Z is empty. In this case, s is a smooth section of a smooth fiber bundle $\mathfrak{X} \to M$, and satisfies the Fueter equation everywhere.

While the compactness theorems and Conjecture 1.4.1 allow for the possibility that Fueter sections appear as limits of ρ–monopoles, they actually do not guarantee that such sections exist. Indeed, one of the major problems in this subject is that of the existence of Fueter sections. To state it more precisely, recall that the Fueter operator $\mathfrak{F}_{\mathbf{p}}$, just as the Seiberg–Witten equation, depends on the choice of a parameter $\mathbf{p} \in \mathscr{P}$. Define $\mathscr{W} \subset \mathscr{P}$ to be the set of those $\mathbf{p} \in \mathscr{P}$ for which there exists a Fueter section (smooth or singular) with respect to $\mathbf{p}$.

In what follows, we assume that the top stratum X of the hyperkähler quotient has positive dimension. If $X = \overline{X} = \{0\}$, as is the case for the classical Seiberg–Witten equation, then the non-compactness phenomenon described in Conjecture 1.4.1 never occurs.

Conjecture 1.4.2 (Codimension one)**.** *$\mathscr{W}$ is a subset of codimension one in $\mathscr{P}$ in the following sense:*

1. *$\mathscr{W}$ is non-empty, closed, and nowhere dense in $\mathscr{P}$.*
2. *For every pair $\mathbf{p}_0, \mathbf{p}_1 \in \mathscr{P} \backslash \mathscr{W}$, a generic path in $\mathscr{P}$ connecting $\mathbf{p}_0$ and $\mathbf{p}_1$ intersects $\mathscr{W}$ at finitely many points. Moreover, around each of these points $\mathscr{W}$ is a codimension one submanifold of $\mathscr{P}$ and the intersection is transverse.*

This prediction is motivated by the following simple example. Suppose that $\overline{X}$ is a quaternionic vector space, so that $\overline{\mathfrak{X}}$ is a vector bundle, and the Fueter operator $\mathfrak{F}_{\mathbf{p}} = \not{D}_{\mathbf{p}}$ is a linear Dirac operator depending on $\mathbf{p} \in \mathscr{P}$. Provided that the parameter space $\mathscr{P}$ is sufficiently large (for example, it consists of all connections on $\overline{\mathfrak{X}}$ compatible with the Clifford multiplication), the set

$$\mathscr{W} = \{\mathbf{p} \in \mathscr{P} \mid \dim\ker \not{D}_{\mathbf{p}} > 0\}$$

has codimension one and separates $\mathscr{P}$ into disjoint chambers. The intersection number of a path $\gamma\colon [0,1] \to \mathscr{P}$ with $\mathscr{W}$ is given by the spectral flow of the family of Dirac operators $(\not{D}_{\gamma(t)})_{t\in[0,1]}$. In many cases, one can compute this spectral flow using the Atiyah–Patodi–Singer theorem [APS76], and prove that $\mathscr{W}$ is non-empty.

For a general, non-linear Fueter operator associated with a hyperkähler quotient, a similar theory of spectral flow has not yet been developed. Nevertheless, one expects the same picture to be true [DS11, Section 6.2]. The Fueter operator $\mathfrak{F}_{\mathbf{p}}$ is a non-linear elliptic operator, whose linearization has index zero. However, solutions to the Fueter equation always come in one-parameter families. Indeed, the hyperkähler quotient $\overline{X} = \mu^{-1}(0)/G$ inherits a rescaling action of $\mathbf{R}^+$ from the representation space S. Thus, there is a free $\mathbf{R}^+$–action on the space of sections of $\mathfrak{X}$, preserving the Fueter equation. As a result, the space of solutions up to rescaling is locally given as the zero set of a Fredholm map of index -1, and, for a sufficiently large class of perturbations $\mathscr{P}$, we expect that a solution exists only for $\mathbf{p} \in \mathscr{P}$ from a codimension one subset.

The above discussion applies only to smooth Fueter sections, which are defined over the entire 3–manifold M. For singular Fueter sections, the situation is much more subtle, as the singular set Z is now part of a solution, and the usual methods of Fredholm theory do not apply. The point is that, given a fixed subset $Z \subset M$, the Fueter equation for a section of $\mathfrak{X}$ over $M\backslash Z$ is not elliptic; its linearization has infinite dimensional cokernel[1]. However, assuming that Z is a C^1–embedded curve which is allowed to move inside the ambient 3–manifold, R. Takahashi [Tak15; Tak17] proved that the deformation theory of pairs (Z, s) up to rescaling is described by a Fredholm problem of index -1. Takahashi considered the case when the fiber of $\overline{\mathfrak{X}}$ is $\mathbf{H}/\mathbf{Z}_2$, the hyperkähler quotient of $\mathbf{H} \otimes \mathfrak{su}(2)$ by $\mathrm{SU}(2)$, which is relevant to the equations studied by Taubes. While Takahashi's work makes significant progress towards establishing Conjecture 1.4.2, the full proof of the conjecture, in addition to considering general hyperkähler quotients, would require dealing with Fueter sections for which the set Z is not smooth. Methods of geometric measure theory are well-suited for this problem, as shown by work of Taubes [Tau14] and B. Zhang [Zha17].

If Conjecture 1.4.1 and Conjecture 1.4.2 are true, then for every $\mathbf{p} \in \mathscr{P}\backslash\mathscr{W}$ the corresponding moduli space of ρ–monopoles is compact. Thus, modulo the usual issues of transversality, reducibles, and orientations, we can define the signed count of ρ–monopoles $n(M, \mathbf{p})$ as in (1.2.2). However, since $\mathscr{W}$ is assumed to be non-empty and of codimension one in $\mathscr{P}$, the set $\mathscr{P}\backslash\mathscr{W}$ may have multiple connected components. For $\mathbf{p}_0, \mathbf{p}_1 \in \mathscr{P}\backslash\mathscr{W}$ from different components, any path connecting $\mathbf{p}_0$ and $\mathbf{p}_1$ might intersect $\mathscr{W}$. The next conjecture describes how each intersection with $\mathscr{W}$ affects the moduli space of ρ–monopoles.

1 We assume here that the section is of Sobolev class $W^{1,2}$, which follows from conditions (3) and (4) in Conjecture 1.4.1. The situation is better when we allow the section, or its derivative, to blow-up along Z, but such Fueter sections do not arise as limits of ρ–monopoles.

Conjecture 1.4.3 (Wall-crossing)**.** *There exists a coorientation on the smooth part of $\mathscr{W}$ such that for every generic pair $\mathbf{p}_0, \mathbf{p}_1 \in \mathscr{P} \backslash \mathscr{W}$ and every generic path $\gamma \colon [0,1] \to \mathscr{P}$ connecting $\mathbf{p}_0$ with $\mathbf{p}_1$, the difference*

$$n(M, \mathbf{p}_1) - n(M, \mathbf{p}_0)$$

is the intersection number of γ with $\mathscr{W}$ with respect to that coorientation.

Apart from the discussion of orientations (which, as we will see in Chapter 3, can be quite subtle) and assuming that the previous two conjectures are proved, Conjecture 1.4.3 boils down to the following analytical result.

Conjecture 1.4.4 (Local wall-crossing)**.** *Suppose that $\mathbf{p} \in \mathscr{W}$ and $\mathscr{W}$ is a codimension one submanifold of $\mathscr{P}$ in a neighborhood of $\mathbf{p}$. Let s be a Fueter section with respect to $\mathbf{p}$.*

Then, for every path $\gamma \colon [-1,1] \to \mathscr{P}$ such that $\gamma(0) = \mathbf{p}$ and γ is transverse to $\mathscr{W}$ at 0, there exist a smooth family $(\varepsilon, \widehat{\Phi}_\varepsilon, A_\varepsilon)$ parametrized by $\varepsilon \in (0, \varepsilon_0]$, and a smooth, monotone function $\varepsilon \mapsto t(\varepsilon)$ such that $t(0) = 0$ and

1. *For every $\varepsilon \in (0, \varepsilon_0]$, the triple $(\varepsilon, \widehat{\Phi}_\varepsilon, A_\varepsilon)$ is a solution to (1.3.2) with respect to the parameter $\gamma(t(\varepsilon))$, so that $(\varepsilon^{-1}\widehat{\Phi}_\varepsilon, A_\varepsilon)$ is a ρ–monopole, i.e. satisfies (1.2.1).*
2. *As $\varepsilon \to 0$, the sequence $(\widehat{\Phi}_\varepsilon, A_\varepsilon)$ converges to the Fueter section s, outside the singular set of s, in the sense described in Conjecture 1.4.1.*

This statement can be seen as the converse to the compactness conjecture. Indeed, Conjecture 1.4.1 asserts that every divergent sequence of ρ–monopoles can be rescaled so that it converges to a Fueter section. Conversely, Conjecture 1.4.4 states that every Fueter section is the limit of a rescaled sequence of ρ–monopoles.

It follows from Conjecture 1.4.4 that the number of ρ–monopoles with respect to the parameter $\gamma(t)$ changes by one as t crosses zero. That is: a ρ–monopole appears or disappears at $t = 0$, depending on whether the reparametrization $\varepsilon \to t(\varepsilon)$ is increasing or decreasing. Thus, the change in the signed count of ρ–monopoles caused by the Fueter section s is:

$$n(M, \gamma(t)) - n(M, \gamma(-t)) = \pm 1,$$

for a small $t > 0$. The sign $\pm$ depends on the function $\varepsilon \to t(\varepsilon)$ and the sign of the ρ–monopole $(\varepsilon^{-1}\widehat{\Phi}_\varepsilon, A_\varepsilon)$ given by the global orientation of the moduli space. A convenient way to package all these signs is by equipping the wall $\mathscr{W}$ with a coorientation, and thus Conjecture 1.4.3 is a global version of Conjecture 1.4.4. A satisfactory solution to this problem should express the intersection number of γ with $\mathscr{W}$ in terms of more familiar, topological or spectral data.

The above conjectures summarize what, in our opinion, are some of the main problems in the study of ρ–monopoles and Fueter sections. The principal difficulty in solving these problems is that the singular set Z of a Fueter section may be a priori very irregular. Optimistically, one would hope that Z must be a finite union of disjoint, smoothly embedded circles, at least for a generic choice of $\mathbf{p} \in \mathscr{P}$. However, proving such a result will most likely require new insights from geometric measure theory and the theory of elliptic differential equations.

In this thesis, we take a different path and develop part of the theory involving only Fueter sections which are smooth (Chapter 2 and Chapter 3) or singular along a 1–dimensional submanifold (Chapter 4). We summarize our main results in Section 1.6, at the end of this chapter. Before doing so, it is worthwhile discussing some of the motivations for studying the Seiberg–Witten equations associated with quaternionic representations. One can envision three different scenarios, in which ρ–monopoles lead to new geometric insights.

Scenario 1. The simplest situation arises when the hyperkähler quotient associated with ρ consists of a single point, i.e. $\overline{X} = \{0\}$. In this case, the moduli space of ρ–monopoles is compact and, if there are no reducibles, the signed count of ρ–monopoles is a topological invariant, as in Example 1.2.2 and Example 1.2.1.

One can consider also higher rank analogues of these examples, corresponding to the choices $G = \mathrm{U}(n)$ or $\mathrm{SU}(n)$ and $S = \mathbf{H} \otimes \mathbf{C}^n$ or $S = \{0\}$. However, in these cases one typically cannot avoid reducibles and one is forced either to consider wall-crossing caused by reducibles, or to develop equivariant Floer theory. For example, a detailed study of reducible solutions to a variant of the $\mathrm{U}(2)$ monopole equation gives a formula relating the Casson invariant and the total Seiberg–Witten invariant of 3–manifolds [Kro98; DG19] . In general, it is expected that $\mathrm{U}(2)$ monopoles can be used to relate instanton Floer homology to its Seiberg–Witten counterpart [KM10, Section 7.9]. This would be a 3–dimensional version of Pidstrygach and Tyurin's approach to Witten's conjecture relating the Donaldson and Seiberg–Witten invariants of 4–manifolds [PT95; FL98b].

A lesser-known equation arises for $G = \mathrm{Sp}(1)$ and $S = \mathbf{H}$, with $\mathrm{Sp}(1)$ acting on $\mathbf{H}$ by right quaternionic multiplication, preserving the left quaternionic structure [Lim03; Ber19]. A peculiar feature of this theory is that the moduli spaces of $\mathrm{Sp}(1)$ monopoles cannot be consistently equipped with orientations using determinant line bundles; see Example 2.4.5. One can still hope to define an invariant, either by counting solutions mod 2, or by building Floer homology groups over $\mathbf{Z}_2$. Walpuski observed that the $\mathrm{Sp}(1)$ Seiberg–Witten equation admits a non-trivial solution whenever M is a hyperbolic 3–manifold.

Scenario 2. While for a general quaternionic representation Fueter sections cannot be ruled out, it might happen that we consider a situation in which there is a distinguished, connected subset of parameters $\mathscr{P}_0 \subset \mathscr{P}$ which intersects $\mathscr{W}$ along a codimension two set. In this case, the number $n(M, \mathbf{p})$ does not depend on the choice of a generic $\mathbf{p} \in \mathscr{P}_0 \backslash \mathscr{W}$.

We will see an example of this situation in Chapter 4, for 3–manifolds of the form $M = S^1 \times \Sigma$ for a surface Σ. In general, if M is a Seifert fibration over Σ, there is a distinguished subset $\mathscr{P}_0 \subset \mathscr{P}$ of parameters compatible with the fibration. In particular, every $\mathbf{p} \in \mathscr{P}_0$ induces a Riemannian metric, and so a complex structure, on Σ, and one expects that the corresponding moduli spaces of ρ–monopoles and Fueter sections can be described in terms of complex-geometric data on Σ. The complex version of Conjecture 1.4.2 is the statement that $\mathscr{P}_0 \cap \mathscr{W}$ is a complex subvariety of $\mathscr{P}_0$ of complex codimension one. In this case, $n(M, \mathbf{p})$ should be independent of the choice of $\mathbf{p} \in \mathscr{P}_0 \backslash \mathscr{W}$ and computable using algebraic geometry. The corresponding 4–dimensional theory of ρ–monopoles on complex surfaces is potentially more interesting, as it might lead to deformation invariants of complex surfaces, cf. [DGP18, Section 4].

There is also a variety of generalized Seiberg–Witten equations which are expected to lead to new topological invariants of links and low-dimensional manifolds, without any wall-crossing. These include the equation for flat $\mathrm{SL}(2, \mathbf{C})$ connections on 3–manifolds [Tau13b], and its 4–dimensional relatives: the Vafa–Witten equation [VW94; Mar10; Tau17] and the Kapustin–Witten equation [KW07;

GU12; MW14; Wit18]. In each of these cases, smooth Fueter sections are easy to understand; for example, in the $SL(2,\mathbf{C})$ story they are simply harmonic 1–forms. However, it is unclear how to rule out singular Fueter sections, and to extract a topological invariant from the moduli spaces of solutions. One can speculate that for each of these equations there is a distinguished family $\mathscr{P}_0$ of perturbations which avoids $\mathscr{W}$. This is known in one situation: following Witten's proposal for a gauge-theoretic construction of the Jones polynomial of a link in S^3, one studies the Kapustin–Witten equation on the half-cylinder $S^3 \times (0,\infty)$ with its natural metric. Since S^3 has positive curvature, the Lichnerowicz–Weitzenböck formula implies that there are no Fueter sections, smooth or singular, and the moduli space of solutions with relevant boundary conditions is compact [Tau18].

There are many interesting interactions between the two pictures—complex and topological—described above. For instance, Tanaka and Thomas [TT17a; TT17b] defined the Vafa–Witten invariants of complex surfaces using algebraic geometry; it would be interesting to find a gauge-theoretic analogue of their construction. In dimension three, flat $SL(2,\mathbf{C})$ connections are related to complex geometry via the Atiyah–Floer picture, in which the moduli space of flat $SL(2,\mathbf{C})$ connections on a 3–manifold is interpreted as the intersection of two complex Lagrangians in the hyperkähler moduli space of flat $SL(2,\mathbf{C})$ connections on a Riemann surface. Using this description and ideas from sheaf theory, Abouzaid and Manolescu [AM19] defined an $SL(2,\mathbf{C})$ analogue of instanton Floer homology.

Scenario 3. In general, there is no distinguished chamber in the space of parameters $\mathscr{P}$, and the the signed count of ρ–monopoles $n(M,\mathbf{p})$ depends on the connected component of $\mathscr{P}\backslash\mathscr{W}$ containing $\mathbf{p}$. In this case, the Seiberg–Witten equation does not lead to a topological invariant. Nevertheless, in the next section we will argue that the the wall-crossing phenomenon described in Conjecture 1.4.3 can still be used to extract some geometric information, for certain choices of the quaternionic representation ρ. While the resulting Seiberg–Witten equation most likely does not have applications in low-dimesional topology, we expect it to play an important role in the study of higher-dimensional manifolds equipped with Riemannian metrics with special holonomy.

1.5 GAUGE THEORY ON G_2–MANIFOLDS

One motivation for studying ρ–monopoles and Fueter sections comes from higher-dimensional Yang–Mills theory. We only briefly outline this relationship as it will not be essential for understanding the results presented in this dissertation. The reader can find more details in the references listed below, especially in [DT98; DS11; Wal13; Don19].

Applications of Yang–Mills theory to low-dimensional geometry and topology rely on the existence of special classes of connections satisfying the Yang–Mills equations: flat connections in dimensions two and three, and instantons in dimension four. These connections obey a simpler, first order differential equation, for which there is a good theory of moduli spaces of solutions.

There is no analogue of the instanton equation on a general higher-dimensional Riemannian manifold. However, such analogues do exist whenever the holonomy group of the Riemannian metric is one of the groups appearing in Berger's list:

$$U(n),\ SU(n),\ Sp(n),\ Sp(n)Sp(1),\ G_2,\ \text{and}\ Spin(7);$$

see [RC98]. The theory of instantons is particularly rich in dimensions six, seven, and eight, for manifolds with holonomy, respectively, $\mathrm{SU}(3)$, G_2, and $\mathrm{Spin}(7)$—one of the groups related to the geometry of octonions. Riemannian 6–manifolds with holonomy $\mathrm{SU}(3)$ are the famous *Calabi–Yau three-folds*, extensively studied in algebraic geometry and string theory. The G_2 and $\mathrm{Spin}(7)$ geometries, which play an important role in a branch of theoretical physics called M–theory, are less understood. There are by now many examples of such manifolds, of which, however, there is still no systematic classification. For more information on Berger's list of holonomy groups and on special holonomy manifolds we refer to the survey articles [Joy04; Bry06; Don19] and textbooks [GHJ03; Joy07].

In the seminal papers [DT98; DS11], Donaldson, Thomas, and E. Segal put forward an idea of defining invariants of Riemannian manifolds with holonomy $\mathrm{SU}(3)$, G_2, and $\mathrm{Spin}(7)$, using the moduli spaces of instantons. For each of the three geometries there is a separate instanton equation, but they are all related by the natural inclusions

$$\mathrm{SU}(3) \subset G_2 \subset \mathrm{Spin}(7).$$

Geometrically, if Z is a Calabi–Yau three-fold, then the cylinder $Z \times \mathbf{R}$ is a G_2–manifold (in the sense that its holonomy group is contained in G_2) and the $\mathrm{SU}(3)$–instanton equation on Z is a dimensional reduction of the G_2–instanton equation on $Z \times \mathbf{R}$. Similarly, if Y is a G_2–manifold, then the cylinder $Y \times \mathbf{R}$ is a $\mathrm{Spin}(7)$–manifold, and the G_2–instanton equation on Y is a dimensional reduction of the $\mathrm{Spin}(7)$–instanton equation on $Y \times \mathbf{R}$. This relationship between the three geometries leads to a conjectural topological field theory picture, analogous to the one known from low dimensions:

dimension	**holonomy**	**type of invariant**
8	$\mathrm{Spin}(7)$	number
7	G_2	vector space
6	$\mathrm{SU}(3)$	category

More precisely, one expects that:

1. For a closed $\mathrm{Spin}(7)$–manifold X there are numerical invariants, analogues of the Donaldson invariants of a 4–manifold, obtained by integrating certain cohomology classes over the moduli spaces of $\mathrm{Spin}(7)$–instantons on X.

2. For a closed G_2–manifold Y there are Floer homology groups obtained by formally applying the construction of Morse homology to a G_2 analogue of the Chern–Simons functional. Critical points of this functional are G_2–instantons on Y, and gradient flow-lines correspond to $\mathrm{Spin}(7)$–instantons on the cylinder $X = Y \times \mathbf{R}$.

3. For every Calabi–Yau three-fold Z there is a category obtained by applying formally the construction of the Fukaya–Seidel category [Sei08; Sei12; Hay14b] to a holomorphic version of the Chern–Simons functional. The construction of this category should involve critical points of this functional, corresponding to $\mathrm{SU}(3)$–instantons on Z, its real flow-lines, corresponding to G_2–instantons on $Y = Z \times \mathbf{R}$, and solutions to the Floer equation, corresponding to $\mathrm{Spin}(7)$–instantons on $X = Z \times \mathbf{R}^2$.

In each case, one should be able to decategorify an invariant to obtain a simpler one, i.e. to obtain a number from a vector space, or a vector space from a category. Thus far, only one of these putative invariants have been rigorously defined: a decategorified version of the SU(3)–invariant. By the Donaldson–Uhlenbeck–Yau theorem [Don85; UY86], SU(3)–instantons on Calabi–Yau three-folds correspond to stable holomorphic vector bundles, which can be studied using methods of algebraic geometry. In particular, while the moduli space of stable bundles is typically non-compact, it embeds into the compact moduli space of semi-stable sheaves, constructed by Gieseker [Gie77] and Maruyama [Mar77; Mar78]. Using deformation theory of sheaves, Thomas [Thoo0] developed a way of extracting a numerical invariant of Calabi–Yau three-folds from the moduli spaces of semi-stable sheaves. This was later refined by various authors to obtain homology groups; see [Sze16] for an introduction to this rich subject. In the last twenty years, the Donaldson–Thomas theory of counting semi-stable sheaves has been a very active area of research in algebraic geometry, with deep connections to representation theory, symplectic topology, and theoretical physics.

In G_2 or Spin(7) geometry, the methods of sheaf theory are unavailable and one tries to define the invariants using analysis. In what follows, we focus on the problem of defining a decategorified, numerical version of the G_2–invariant. The most naive approach would be to simply count G_2–instantons with signs, in the same way one defines the Casson invariant of a 3–manifold as a signed count of flat connections [Tau90]. More precisely, let Y be a compact G_2–manifold. The exceptional Lie group G_2 can be defined as the stabilizer in SO(7) of a generic 3–form in $\Lambda^3\mathbf{R}^7$. It follows that Y is equipped with a parallel (in particular, closed) 3–form ϕ. A G_2–instanton is a connection A on a principal bundle over Y whose curvature form F_A satisfies a 7–dimensional analogue of the instanton equation:

$$*(F_A \wedge \phi) = -F_A.$$

The deformation theory of G_2–instantons up to gauge equivalence is governed by an elliptic operator of index zero. Thus, for a generic perturbation of the equation, the moduli space of solutions consists of isolated points. One natural way of perturbing the equation is to relax the condition for Y to have a Riemannian metric of holonomy G_2, and to consider instead *tamed almost G_2–structures* introduced in [DS11, Section 3]; one can use also holonomy perturbations. In any case, suppose that we have a sufficiently large space of perturbations $\mathscr{P}_Y$ such that for a generic choice of $\mathbf{p} \in \mathscr{P}_Y$, the moduli space of perturbed G_2–instantons consists of isolated points. There is a natural way of prescribing signs to these points [Wal13, Section 6.1], [JU19], and a naive definition of the invariant is

$$N_0(Y, \mathbf{p}) = \sum_{G_2 \text{ instantons } [A]} \operatorname{sign}(A), \tag{1.5.1}$$

where the sum is taken over all gauge equivalence classes of G_2–instantons on a fixed principal bundle on Y.

The main problem with this definition is that the moduli space of G_2–instantons might be non-compact, in which case (1.5.1) is an infinite sum. Even if the moduli space is compact for a generic choice of $\mathbf{p} \in \mathscr{P}_Y$, it can still happen that for a path $\gamma\colon [0,1] \to \mathscr{P}$ connecting two such $\mathbf{p}_0$, $\mathbf{p}_1$, the moduli space with respect to $\gamma(t)$ ceases to be compact for some $t \in (0,1)$. If this is the case, we cannot conclude from the usual cobordism argument that $N_0(Y, \mathbf{p}_0) = N_0(Y, \mathbf{p}_1)$, and the number (1.5.1) fails to define an invariant of the G_2–manifold Y.

The non-compactness of the moduli space is caused by the phenomenon of bubbling, discovered by Uhlenbeck, which occurs when a sequence of G_2–instantons concentrates along a closed 3–dimensional subset $M \subset Y$, developing along M a singularity in a way familiar from the study of instantons on 4–manifolds [Uhl82; Pri83; Nak88; TT04]. Tian [Tia00] proved that M is a rectifiable 3–current and, if it is actually a smooth 3–dimensional submanifold, then it is *associative* in the sense that the restriction of ϕ to M is equal to the volume form of M. Such submanifolds are calibrated in the sense of Harvey and Lawson [HL82]; in particular, they are minimal submanifolds whose volume depends only on the homology class. Moreover, their deformation theory is governed by an elliptic operator of index zero, so we expect that generically there are only finitely many such submanifolds within a given homology class.

However, not every associative submanifold $M \subset Y$ can be the bubbling locus of a sequence of G_2–instantons. By analyzing the G_2–instanton equation in a tubular neighborhood of such a submanifold, Donaldson and Segal conjectured that M is a bubbling locus if and only if there exists a certain Fueter section on M [DS11, Section 6.1], [Hay12], [Wal17, Section 4]. This Fueter section takes values in the fiber bundle $\mathfrak{X} \to M$ whose fiber over a point $x \in M$ is the moduli space of instantons on $N_x = \mathbf{R}^4$, the normal space to M at x. The idea is that such a Fueter sections is a model for a G_2–instanton in a tubular neighborhood of M, which is highly concentrated along M.

The Fueter operator acting on the space of section of $\mathfrak{X}$ depends on various parameters, including the Riemannian metric on M induced from the embedding $M \subset Y$, which, in turn, depend on the choice of $\mathbf{p} \in \mathscr{P}_Y$. Thus, we have the following three predictions analogous to the conjectures from the previous section:

1. *Compactness.* (cf. Conjecture 1.4.1) Given a sequence $\mathbf{p}_i \in \mathscr{P}_Y$ converging to $\mathbf{p}$, and a sequence A_i of G_2–instantons with respect to $\mathbf{p}_i$, which bubbles along an associative submanifold $M \subset Y$, there exists a Fueter section of the instanton bundle $\mathfrak{X} \to M$ with respect to $\mathbf{p}$.

2. *Codimension one.* (cf. Conjecture 1.4.2) The set $\mathscr{W}_Y$ of those $\mathbf{p} \in \mathscr{P}_Y$ for which there exists an associative submanifold $M \subset Y$ and a Fueter section of $\mathfrak{X} \to M$, has codimension one in $\mathscr{P}_Y$. In particular, for a generic choice of $\mathbf{p} \in \mathscr{P}_Y$, the moduli space of G_2–instantons is finite and the signed count $N_0(Y, \mathbf{p})$ of G_2–instantons, as in (1.5.1), is well-defined.

3. *Wall-crossing.* (cf. Conjecture 1.4.3) When a path $\gamma\colon [0,1] \to \mathscr{P}_Y$ intersects the codimension one set $\mathscr{W}_Y$, then $N_0(Y, \gamma(t))$ changes by ± 1.

In reality, one has to consider also singular associatives and singular Fueter sections (we will come back to this point). Even under the assumption that M is smooth, the first conjecture has not yet been proved, which is a major gap in this theory. A version of the third conjecture was proved by Walpuski [Wal17].

If these predictions are correct, then $N_0(Y, \mathbf{p})$, defined by (1.5.1), is not an invariant of the G_2–manifold Y as it can jump when we vary $\mathbf{p}$. Walpuski [Wal13; Wal17] proposed to compensate for such jumps by adding to (1.5.1) the signed count of certain ρ–monopoles on M. The quaternionic representation ρ is chosen so that the associated hyperkähler quotient is, via the famous ADHM construction [Ati+78], the moduli space of anti-self-dual instantons on $\mathbf{R}^4$, and so in this case ρ–monopoles are called *ADHM monopoles*. As explained in the previous section, the signed count of ADHM monopoles on M is not a topological invariant of M; it can jump when we deform the Riemannian metric on M and other parameters of the ADHM Seiberg–Witten equation. The point is that such jumps should occur exactly when there exists a Fueter section of the instanton bundle $\mathfrak{X} \to M$. Thus, Walpuski proposed a new definition of the putative G_2–invariant, which takes under account not only G_2–instantons but also associative submanifolds and ADHM monopoles. Schematically,

$$N(Y,\mathbf{p}) = \underbrace{\sum_{G_2 \text{ instantons } [A]} \operatorname{sign}(A)}_{N_0(Y,\mathbf{p})} + \sum_{\text{associatives } M} n(M,\mathbf{p}) \tag{1.5.2}$$

Here, the sum is taken over all gauge equivalence classes of G_2–instantons on a fixed principal bundle, and all associatives in Y within a certain finite range of homology classes depending on the Chern classes of the bundle; $n(M,\mathbf{p})$ is the signed count of ADHM monopoles on M, as in (1.2.2), with respect to the parameters, such as Riemannian metric etc., induced from $\mathbf{p} \in \mathscr{P}_Y$ via the embedding $M \subset Y$. We refer to [DW17b, Section 6] for more details.

The wall-crossing conjectures for G_2–instantons and ADHM monopoles would imply the invariance of $N(Y,\mathbf{p})$ under deformations of $\mathbf{p}$. Indeed, given a path in $\mathscr{P}_Y$ which starts at $\mathbf{p}_0$, crosses the wall $\mathscr{W}_Y$ exactly once, and ends at $\mathbf{p}_1$, we would know that

$$N_0(Y,\mathbf{p}_1) - N_0(Y,\mathbf{p}_0) = \pm 1, \quad \text{and}$$
$$n(M,\mathbf{p}_1) - n(M,\mathbf{p}_0) = \pm 1,$$

where M is the bubbling locus of the sequence of G_2–instantons which concentrates as the path crosses $\mathscr{W}_Y$. (We assume here that the path is short and the associative M is unobstructed, so that it deforms inside Y as we vary $\mathbf{p}$.) Thus, modulo the question of orientations, the number $N(Y,\mathbf{p})$ remains unchanged.

There are various difficulties in making this proposal rigorous. Far from being merely technical obstacles, these difficulties reflect important gaps in our understanding of calibrated geometry and gauge theory, cf. [Don19, Section 3.5]

1. *Singularities of associatives.* In general, we cannot guarantee that the bubbling locus M of a sequence of G_2–instantons is a smooth submanifold. One would optimistically hope that the bubbling locus is smooth for a generic choice of $\mathbf{p}$ (for example, for a generic tamed almost G_2 structure on Y). Even then, associative submanifolds might form singularities as we vary $\mathbf{p}$ in a one-parameter family. It is an important question in G_2 geometry to classify all generic singularities of one-parameter families of associatives [Joy18]. Such singularities can lead to a change in the number and topology of associatives, and, as a result, to possible wall-crossing phenomena for the terms appearing in (1.5.2). However, it is possible that the sum of all the terms is invariant under these transitions because of the gluing and surgery formulae for the signed count of ADHM monopoles [DW17b].

2. *Singularities of G_2–instantons.* In addition to bubbling, Uhlenbeck's compactness theorem allows for the possibility that a sequence of G_2–instantons develops singularities of codimension greater than four. This could lead to interactions between smooth G_2–instantons and G_2–instantons with singularities; understanding such interactions requires developing a theory

of deforming and smoothing singular G_2–instantons, cf. [Wan16]. Even if singular G_2–instantons remain isolated from the smooth ones, one can envision a scenario in which a singular Fueter section on an associative $M \subset Y$ gives rise to a singular G_2–instanton, via a gluing construction similar to the one in [Wal17]. As this process would be accompanied by a change in the number of ADHM monopoles, the sum (1.5.2) would have to include also a signed count of singular G_2–instantons in order to remain invariant.

3. *Reducible ADHM monopoles.* If M is a homology 3–sphere, then there is an additional wall-crossing phenomenon for the signed count of ADHM monopoles $n(M, \mathbf{p})$, caused by reducible solutions. Unless one can guarantee that there are no associatives homology 3–spheres inside Y, it is possible that the G_2–invariant (1.5.2) cannot be defined at all and one has to construct a categorified version of the invariant. This problem is familiar from gauge theory on 3–manifolds, where it can be solved within the framework of equivariant Floer homology. (Floer homology groups of homology 3–spheres are infinitely generated and so one cannot define a decategorified, numerical invariant by taking the rank of the group.) A candidate for G_2–Floer homology would be the homology of a complex generated by G_2–instantons, associatives, and ADHM monopoles, with differential counting Spin(7)–instantons, Cayley cobordisms, and 4–dimensional ADHM monopoles, inside the cylinder $X = Y \times \mathbf{R}$ [DW17b].

We hope that this brief and rather speculative discussion convinces the reader that defining enumerative invariants of G_2–manifolds is an exciting and challenging project. Completing it will require a deeper understanding of G_2–instantons, associative submanifolds, and ρ–monopoles. The goal of this dissertation is to make a contribution to the third subject, in particular by studying the relationship between ρ–monopoles and Fueter sections.

1.6 SUMMARY OF THE RESULTS

Chapter 2: Monopoles and Fueter sections

This chapter is based on the articles [DW17a] and [DW17b, Section 5] written in collaboration with Thomas Walpuski.

After reviewing the general theory of ρ–monopoles and Fueter sections, we prove the main theorems in Section 2.7 and Section 2.8. These theorems establish Conjecture 1.4.4 under the assumption that the Fueter section s is smooth. In the final Section 2.2, we introduce the Seiberg–Witten equations associated with the quaternionic representations appearing in the ADHM construction of instantons. As explained earlier, solutions to this equation, ADHM monopoles, are expected to play an important role in defining enumerative invariants of G_2–manifolds.

Chapter 3: Fueter sections with singularities

This chapter is based on the article [DW17c] written in collaboration with Thomas Walpuski, and incorporates also a small part of [Doa17b].

We extend the results of the previous chapter to develop a theory of counting *monopoles with two spinors*. These are solutions to the Seiberg–Witten equation with two spinors, one of the ADHM Seiberg–Witten equations[2]. The main result is a wall-crossing formula, Theorem 3.5.7, which proves Conjecture 1.4.3 for this equation under the assumption that there exist no singular Fueter sections along the path γ. As an application, we prove Theorem 3.1.7, which asserts the existence of singular Fueter sections with values in $\mathbf{H}/\mathbf{Z}_2$ on every 3–manifold M with $b_1(M) > 1$. Such singular Fueter sections are examples of *harmonic* $\mathbf{Z}_2$ *spinors* defined by Taubes [Tau14]. In particular, Theorem 3.1.7 produces the first examples of singular harmonic $\mathbf{Z}_2$ spinors which are not obtained by means of complex geometry. In Section 3.11 we discuss the significance of our wall-crossing formula to gauge theory on G_2–manifolds.

Chapter 4: ADHM monopoles on Riemann surfaces

This chapter is based on [Doa17b; Doa17a], apart from the Section 4.2, which is taken from [DW17b, Section 7.4], written in collaboration with Thomas Walpuski.

The discussion of Chapter 2 and Chapter 3 shows that a complete theory of counting ADHM monopoles must incorporate singular Fueter sections and compactifications of the moduli spaces. While such a general theory is yet to be developed, the purpose of Chapter 4 is to focus on examples. We do this by studying ADHM monopoles on Riemannian 3–manifolds of the form $M = S^1 \times \Sigma$ for a Riemann surface Σ. We show that in this case all ADHM monopoles are pulled back from Σ and correspond to certain holomorphic data on Σ.

In the special case of the Seiberg–Witten equation with two spinors, we construct two compactifications of the moduli space, one using gauge theory and one using algebraic geometry, and we prove that they are homeomorphic. Two aspects of this result shed light on the general problem of compactifying the moduli space for an arbitrary 3–manifold. First, our construction involves refining the compactness theorem of Haydys and Walpuski [HW15] in the case $M = S^1 \times \Sigma$. Second, we show that there are topological and analytical obstructions for a harmonic $\mathbf{Z}_2$ spinor to appear in the compactification. As a corollary, we construct examples of harmonic $\mathbf{Z}_2$ spinors which are not limits of any sequence of monopoles with two spinors. Our discussion of compactness provides a model of what should be proved for a general 3–manifold. Note, in particular, that for any 3–manifold M, the behaviour of a sequence of monopoles with two spinors converging to a harmonic $\mathbf{Z}_2$ spinor singular along $Z \subset M$ is in a small neighborhood of Z modelled on the S^1–invariant situation described here.

Finally, we show that for a generic choice of S^1–invariant parameters of the Seiberg–Witten equation with two spinors over $M = S^1 \times \Sigma$, harmonic $\mathbf{Z}_2$ spinors do not exist and the moduli space is a compact Kähler manifold. After a perturbation it splits into isolated points which can be counted with signs, yielding a number independent of the initial choice of the S^1–invariant parameters. This shows that the number $n(M, \mathbf{p})$ is well-defined in this case and independent of $\mathbf{p}$, as long as $\mathbf{p}$ is pulled back from Σ. We compute this number for surfaces of low genus and construct many explicit examples of monopoles with two spinors and harmonic $\mathbf{Z}_2$ spinors using algebraic geometry.

2 In general, there is an ADHM Seiberg–Witten equation for every pair (k, n) corresponding to the charge and rank of instantons on $\mathbf{R}^4$. The classical Seiberg–Witten equation corresponds to $k = 1$, $n = 1$, while the Seiberg–Witten equation with two spinors corresponds to $k = 1$, $n = 2$.

2 MONOPOLES AND FUETER SECTIONS

A key difficulty in studying generalized Seiberg–Witten equations arises from the non-compactness issue caused by a lack of a priori bounds on the spinor. This phenomenon has been studied in special cases by Taubes [Tau13b; Tau13a; Tau16], and Haydys and Walpuski [HW15]. To focus on the issue of the spinor becoming very large, one passes to a blown-up Seiberg–Witten equation. The lack of a priori bounds then manifests itself as the equation becoming degenerate elliptic when the norm of the spinor tends to infinity. However, a theorem of Haydys allows us to reinterpret the limiting equation as a non-linear version of the Dirac equation, known as the Fueter equation.

In this chapter, we review the general theory of Seiberg–Witten and Fueter equations and develop Fredholm deformation theory describing the local Kuranishi structure of moduli spaces of solutions to both equations. The main result, proved in Section 2.7, asserts that the two deformation theories can be glued together, yielding a Kuranishi space with boundary, whose interior consists of monopoles, solutions to Seiberg–Witten equations, and boundary of Fueter sections, up to rescaling. This leads to a description, in Section 2.8, of a wall-crossing phenomenon for monopoles, whose signed count can change when we vary parameters of a Seiberg–Witten equation.

Finally, we discuss the Seiberg–Witten equations associated with the quaternionic representations appearing in the ADHM construction of instantons. Solutions to these equations, ADHM monopoles, play a crucial role in the Haydys–Walpuski program outlined in the introduction.

REFERENCES. Apart from some minor changes, the content of this chapter is the same as that of the article [DW17a]. The discussion of ADHM monopoles follows [DW17b, Section 5]. Both of these articles were written in collaboration with Thomas Walpuski.

2.1 HYPERKÄHLER QUOTIENTS OF QUATERNIONIC REPRESENTATIONS

Definition 2.1.1. Let $\mathbf{H} = \mathbf{R}\langle 1, i, j, k\rangle$ be the division algebra of quaternions. Denote by $\operatorname{Im}\mathbf{H} = \mathbf{R}\langle i, j, k\rangle$ the subspace of imaginary quaternions.

A *quaternionic Hermitian vector space* is a real vector space S together with a linear map $\gamma\colon \operatorname{Im}\mathbf{H} \to \operatorname{End}(S)$ and a Euclidean inner product $\langle\cdot,\cdot\rangle$ such that γ makes S into a left module over $\mathbf{H}$, and i, j, k act by isometries. The *unitary symplectic group* $\operatorname{Sp}(S)$ is the subgroup of $\operatorname{GL}(S)$ preserving γ and $\langle\cdot,\cdot\rangle$.

Let G be a compact, connected Lie group.

Definition 2.1.2. A *quaternionic representation* of G is a Lie group homomorphism $\rho\colon G \to \operatorname{Sp}(S)$ for some quaternionic Hermitian vector space S.

Suppose that a quaternionic representation $\rho\colon G \to \operatorname{Sp}(S)$ has been chosen. By slight abuse of notation, we also denote the induced Lie algebra representation by

$\rho\colon\ \mathfrak{g} \to \mathfrak{sp}(V)$. We combine ρ and γ into the map $\bar{\gamma}\colon\ \mathfrak{g} \otimes \operatorname{Im}\mathbf{H} \to \operatorname{End}(S)$ defined by

$$\bar{\gamma}(\xi \otimes v)\Phi := \rho(\xi)\gamma(v)\Phi.$$

The map $\bar{\gamma}$ takes values in symmetric endomorphisms of S. Denote the adjoint of $\bar{\gamma}$ by $\bar{\gamma}^*\colon\ \operatorname{End}(S) \to (\mathfrak{g} \otimes \operatorname{Im}\mathbf{H})^*$.

Proposition 2.1.3. *The map $\mu\colon\ S \to (\mathfrak{g} \otimes \operatorname{Im}\mathbf{H})^*$ defined by*

$$\mu(\Phi) := \frac{1}{2}\bar{\gamma}^*(\Phi\Phi^*) \tag{2.1.1}$$

with $\Phi^* := \langle \Phi, \cdot \rangle$ is a hyperkähler moment map, that is, it is G–equivariant, and

$$\langle (d\mu)_\Phi \phi, \xi \otimes v \rangle = \langle \gamma(v)\rho(\xi)\Phi, \phi \rangle$$

for all $\xi \in \mathfrak{g}$ and $v \in \operatorname{Im}\mathbf{H}$.

This is a straightforward calculation. Nevertheless, it leads to an important conclusion: there is a hyperkähler orbifold naturally associated with the quaternionic representation.

Definition 2.1.4. We call $\Phi \in S$ *regular* if $(d\mu)_\Phi\colon\ T_\Phi S \to (\mathfrak{g} \otimes \operatorname{Im}\mathbf{H})^*$ is surjective. Denote by S^{reg} the open cone of regular elements of S.

By Hitchin et al. [Hit+87, Section 3(D)], the *hyperkähler quotient*

$$X := S^{\mathrm{reg}} /\!/\!/ G := \left(\mu^{-1}(0) \cap S^{\mathrm{reg}}\right) / G$$

is a hyperkähler orbifold. For convenience, in this dissertation we will always assume that X is, in fact, a hyperkähler manifold, i.e. that G acts freely on $\mu^{-1}(0) \cap S^{\mathrm{reg}}$. It will be important later that X is a cone; that is, it carries an isometric free $\mathbf{R}^+$–action induced from the scalar multiplication on S.

For future reference, let us recall some technical details of the construction and main properties of the hyperkähler quotient.

Proposition 2.1.5 (Hitchin et al. [Hit+87, Section 3(D)]). *If $\rho\colon\ G \to \operatorname{Sp}(S)$ is a quaternionic representation such that G acts freely on $\mu^{-1}(0) \cap S^{\mathrm{reg}}$, then the following hold:*

1. *The space $X = S^{\mathrm{reg}} /\!/\!/ G$ is manifold.*
2. *Denote by $p\colon\ \mu^{-1}(0) \cap S^{\mathrm{reg}} \to X$ the canonical projection. Set*

   $$\mathfrak{H} := (\ker \mathrm{d}p)^\perp \cap T(\mu^{-1}(0) \cap S^{\mathrm{reg}}) \quad \textit{and}$$
   $$\mathfrak{N} := \mathfrak{H}^\perp \subset TS|_{\mu^{-1}(0)\cap S^{\mathrm{reg}}}.$$

   For each $\Phi \in \mu^{-1}(0) \cap S^{\mathrm{reg}}$, $(\mathrm{d}p)_\Phi\colon\ \mathfrak{H}_\Phi \to T_{[\Phi]}X$ is an isomorphism, and

   $$\mathfrak{N}_\Phi = \operatorname{im}\left(\rho(\cdot)\Phi \oplus \bar{\gamma}(\cdot)\Phi\colon\ \mathfrak{g} \otimes \mathbf{H} \to S\right). \tag{2.1.2}$$

3. *For each $\Phi \in \mu^{-1}(0) \cap S^{\mathrm{reg}}$, γ preserves the splitting $S = \mathfrak{H}_\Phi \oplus \mathfrak{N}_\Phi$.*

4. *There exist a Riemannian metric g_X on X and a Clifford multiplication*

$$\gamma_X\colon\ \operatorname{Im}\mathbf{H} \to \operatorname{End}(TX)$$

such that

$$p^* g_X = \langle\cdot,\cdot\rangle \quad \textit{and} \quad p^*\gamma_X = \gamma.$$

5. *γ_X is parallel with respect to g_X; hence, X is a hyperkähler orbifold—which is called the* hyperkähler quotient *of S by G.*

Remark 2.1.6. More generally, $\mu^{-1}(0)/G$ can be decomposed into a union of hyperkähler manifolds according to the conjugacy class of the stabilizers in G; see Dancer and Swann [DS97, Theorem 2.1].

The following summarizes the algebraic data required to write a Seiberg–Witten equation.

Definition 2.1.7. A set of *algebraic data* consists of:

1. a quaternionic Hermitian vector space $(S, \gamma, \langle\cdot,\cdot\rangle)$,
2. a compact, connected Lie group G together with an Ad–invariant inner product on the Lie algebra $\mathfrak{g}$ of G,
3. a quaternionic representation $\rho\colon\ G \to \operatorname{Sp}(S)$ such that G acts freely on $\mu^{-1}(0) \cap S^{\mathrm{reg}}$.

While the group G and its quaternionic representation ρ are the main objects of interest, in what follows, it will be convenient to consider also the following as part of algebraic data:

4. a compact, connected Lie group H containing G as a normal subgroup, together with an Ad–invariant inner product on the Lie algebra $\mathfrak{h}$ of H extending the inner product on $\mathfrak{g}$,
5. an extension of $\rho\colon\ G \to \operatorname{Sp}(S)$ to a quaternionic representation $H \to \operatorname{Sp}(S)$ (which, for simplicity, we will also denote by ρ).

Given such algebraic data, the group $K = H/G$ is called the *flavor symmetry group*.

The groups G and K play different roles: G is the structure group of the equation, whereas K consists of any additional symmetries, which can be used to twist the setup or remain as symmetries of the theory. On first reading, the reader should feel free to assume for simplicity that $H = G \times K$, or even that $H = G$ and K is trivial.

2.2 SEIBERG–WITTEN EQUATIONS

Let M be a closed, connected, oriented 3–manifold.

Definition 2.2.1. A set of *geometric data* on M compatible with a set of algebraic data as in Definition 2.1.7 consists of:

1. a Riemannian metric g,

2. a spin structure $\mathfrak{s}$, thought of as a principal $\mathrm{Sp}(1)$–bundle $\mathfrak{s} \to M$ together with an isomorphism $T^*M \cong \mathfrak{s} \times_{\mathrm{Sp}(1)} \operatorname{Im} \mathbf{H}$,
3. a principal H–bundle $Q \to M$, and
4. a connection B on the principal K–bundle

$$R := Q \times_H K.$$

Remark 2.2.2. The following observation is due to Haydys [Hay14a, Section 3.1]. Suppose there is a homomorphism $\mathbf{Z}_2 \to Z(H)$ such that the non-unit in $\mathbf{Z}_2$ acts through ρ as -1. Set $\hat{H} := (\mathrm{Sp}(1) \times H)/\mathbf{Z}_2$. All of the constructions in this section go through with $\mathfrak{s} \times Q$ replaced by a $\hat{H}$–principal bundle $\hat{Q}$. This generalization is important when formulating the Seiberg–Witten equation in dimension four, since not every oriented 4–manifold admits a spin structure. (Recall that every oriented 3–manifold admits a spin structure.) In the classical Seiberg–Witten theory, this corresponds to endowing the manifold with a spin^c structure rather than a spin structure and a $U(1)$–bundle.

Suppose that a set of geometric data as in Definition 2.2.1 has been fixed. Left-multiplication by unit quaternions defines an action $\theta\colon \mathrm{Sp}(1) \to \mathrm{O}(S)$ such that

$$\theta(q)\gamma(v)\Phi = \gamma(\mathrm{Ad}(q)v)\theta(q)\Phi$$

for all $q \in \mathrm{Sp}(1) = \{q \in \mathbf{H} : |q| = 1\}$, $v \in \operatorname{Im} \mathbf{H}$, and $\Phi \in S$. This can be used to construct various vector bundles and operators as follows.

Definition 2.2.3. The *spinor bundle* is the vector bundle

$$\mathfrak{S} := (\mathfrak{s} \times Q) \times_{\mathrm{Sp}(1) \times H} S.$$

Since $T^*M \cong \mathfrak{s} \times_{\mathrm{Sp}(1)} \operatorname{Im} \mathbf{H}$, it inherits a *Clifford multiplication* $\gamma\colon\ T^*M \to \mathrm{End}(\mathfrak{S})$.

Definition 2.2.4. Denote by $\mathscr{A}(Q)$ the space of connections on Q. Set

$$\mathscr{A}_B(Q) := \{A \in \mathscr{A}(Q) : A \text{ induces } B \text{ on } R\}.$$

$\mathscr{A}_B(Q)$ is an affine space modeled on $\Omega^1(M, \mathfrak{g}_P)$ with $\mathfrak{g}_P$ denoting the *adjoint bundle* associated with $\mathfrak{g}$, that is,

$$\mathfrak{g}_P := Q \times_{\mathrm{Ad}} \mathfrak{g}.$$

If the structure group of the bundle Q can be reduced from H to $G \lhd H$, then $\mathfrak{g}_P$ is the adjoint bundle of the resulting principal G–bundle P; hence the notation. For example, if $H = G \times K$, then such a reduction P exists. In general, P might not exist but traces of it do—e.g., its adjoint bundle and its gauge group.

Definition 2.2.5. Every $A \in \mathscr{A}_B(Q)$ defines a covariant derivative $\nabla_A\colon\ \Gamma(\mathfrak{S}) \to \Omega^1(M, \mathfrak{S})$. The *Dirac operator* associated with A is the linear map

$$\not{D}_A\colon\ \Gamma(\mathfrak{S}) \to \Gamma(\mathfrak{S})$$

defined by

$$\not{D}_A \Phi := \gamma(\nabla_A \Phi),$$

where $\gamma\colon\ T^*M \otimes \mathfrak{S} \to \mathfrak{S}$ is the bundle version of the Clifford multiplication.

Definition 2.2.6. The hyperkähler moment map $\mu\colon S \to (\operatorname{Im}\mathbf{H} \otimes \mathfrak{g})^*$ induces a map

$$\mu\colon \mathfrak{S} \to \Lambda^2 T^*M \otimes \mathfrak{g}_P$$

since, using the Riemannian metric, $(T^*M)^* \cong \Lambda^2 T^*M$.

Denoting by

$$\varpi\colon \mathfrak{g}_Q \to \mathfrak{g}_P$$

the projection induced by $\mathfrak{h} \to \mathfrak{g}$, we are finally in a position to state the equation we wish to study.

Definition 2.2.7. The *Seiberg–Witten equation* associated with the chosen geometric data is the following system of differential equations for $(\Phi, A) \in \Gamma(\mathfrak{S}) \times \mathscr{A}_B(Q)$:

$$\begin{aligned} \not{D}_A\Phi &= 0 \quad \text{and} \\ \varpi F_A &= \mu(\Phi). \end{aligned} \tag{2.2.1}$$

Definition 2.2.8. A *ρ–monopole* is a solution (Φ, A) of the Seiberg–Witten equation (2.2.1) associated with a quaternionic representation $\rho\colon G \to \mathrm{Sp}(S)$, or, more precisely, with a choice of algebraic data as in Definition 2.1.7 and compatible geometric data as in Definition 2.2.1. In a context in which this data is fixed, we will call (Φ, A) simply a *monopole*.

Most of the well-known equations of mathematical gauge theory on 3– and 4–manifolds can be obtained as a Seiberg–Witten equation.[1]

Example 2.2.9. $S = \mathbf{H}$ and $\rho\colon \mathrm{U}(1) \to \mathbf{H}$ acting by right-multiplication with $e^{i\theta}$ leads to the *classical Seiberg–Witten equation* in dimension three.

Example 2.2.10. Let $G = \mathrm{U}(n)$ and $S = \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^n$, where the complex structure on $\mathbf{H}$ is given by right-multiplication by i. Let $\rho\colon \mathrm{U}(n) \to \mathrm{Sp}(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^n)$ be induced from the standard representation of $\mathrm{U}(n)$. The corresponding Seiberg–Witten equation is the *$\mathrm{U}(n)$–monopole equation* in dimension three. The closely related $\mathbf{P}\mathrm{U}(2)$–monopole equation on 4–manifolds plays a crucial role in Pidstrigach and Tyurin's approach to proving Witten's conjecture relating Donaldson and Seiberg–Witten invariants; see, e.g., [PT95; FL98a; Tel00].

In both of the above examples we have $\mu^{-1}(0) = \{0\}$ and the corresponding hyperkähler quotient $X = S^{\mathrm{reg}} /\!/\!/ G$ is empty. As we will see in Section 2.5, the condition $\mu^{-1}(0) = \{0\}$ implies compactness for the corresponding moduli spaces of solutions; this observation is due to Witten.

Example 2.2.11. Let G be a compact Lie group, $\mathfrak{g} = \mathfrak{g}$, and fix an Ad–invariant inner product on $\mathfrak{g}$. $S := \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{g}$ is a quaternionic Hermitian vector space, and $\rho\colon G \to \mathrm{Sp}(S)$ induced by the adjoint action is a quaternionic representation. The moment map $\mu\colon \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{g} \to (\operatorname{Im}\mathbf{H} \otimes \mathfrak{g})^*$ is given by

$$\begin{aligned} \mu(\xi) &= \frac{1}{2}[\xi, \xi] \\ &= ([\xi_2, \xi_3] + [\xi_0, \xi_1]) \otimes i + ([\xi_3, \xi_1] + [\xi_0, \xi_2]) \otimes j + ([\xi_1, \xi_2] + [\xi_0, \xi_3]) \otimes k \end{aligned}$$

1 In fact, if we allow the Lie groups and the representations to be infinite-dimensional, we can also recover (special cases of) the G_2– and Spin(7)–instanton equations [Hay12, Section 4.2].

for $\xi = \xi_0 \otimes 1 + \xi_1 \otimes i + \xi_2 \otimes j + \xi_3 \otimes k \in \mathbf{H} \otimes_{\mathbf{R}} \mathfrak{g}$. Set $H := \mathrm{Sp}(1) \times G$ and extend the above quaternionic representation of G to H by declaring that $q \in \mathrm{Sp}(1)$ acts by right-multiplication with q^*.

Taking Q to be the product of the chosen spin structure $\mathfrak{s}$ with a principal G–bundle, and choosing B such that it induces the spin connection on $\mathfrak{s}$, (2.2.1) becomes

$$\begin{aligned} \mathrm{d}_A^* a &= 0, \\ *\mathrm{d}_A a + \mathrm{d}_A \xi &= 0, \quad \text{and} \\ F_A &= \tfrac{1}{2}[a \wedge a] + *[\xi, a]. \end{aligned}$$

for $\xi \in \Gamma(\mathfrak{g}_P)$, $a \in \Omega^1(M, \mathfrak{g}_P)$ and $A \in \mathscr{A}(P)$. If M is closed, then integration by parts shows that every solution of this equation satisfies $\mathrm{d}_A \xi = 0$ and $[\xi, a] = 0$; hence, $A + ia$ defines a *flat $G^{\mathbf{C}}$–connection*. Here $G^{\mathbf{C}}$ denotes the complexification of G.

In the above situation, we have $\mu^{-1}(0)/G \cong (\mathbf{H} \otimes \mathfrak{t})/W$ where $\mathfrak{t}$ is a the Lie algebra of a maximal torus $T \subset G$ and $W = N_G(T)/T$ is the Weyl group of G. However, since each $\xi \in \mu^{-1}(0)$ has stabilizer conjugate to T, we have $\mu^{-1}(0) \cap S^{\mathrm{reg}} = \varnothing$, and the hyperkähler quotient $S^{\mathrm{reg}} /\!/\!/ G$ is empty.

The Seiberg–Witten equation is invariant with respect to gauge transformations which preserve the flavor bundle R.

Definition 2.2.12. The *group of restricted gauge transformations* is

$$\mathscr{G}(P) := \{u \in \mathscr{G}(Q) : u \text{ acts trivially on } R\}.$$

$\mathscr{G}(P)$ is an infinite dimensional Lie group with Lie algebra $\Omega^0(M, \mathfrak{g}_P)$. If Q has a reduction to a principal G–bundle P, then $\mathscr{G}(P)$ is isomorphic to the gauge group of P.

The restricted gauge group $\mathscr{G}(P)$ acts on $\Gamma(\mathfrak{S}) \times \mathscr{A}_B(Q)$ preserving the space of solutions of (2.2.1). The main object of our study is the space of solutions to (2.2.1) modulo restricted gauge transformations. This space depends on the geometric data chosen as in Definition 2.2.1. The topological part of the data, the bundles $\mathfrak{s}$ and H, will be fixed. The remaining parameters of the equations, the metric g and the connection B, will be allowed to vary.

Definition 2.2.13. Let $\mathscr{M}et(M)$ be the space of Riemannian metrics on M. The *parameter space* is

$$\mathscr{P} := \mathscr{M}et(M) \times \mathscr{A}(R).$$

Definition 2.2.14. For $\mathbf{p} = (g, B) \in \mathscr{P}$, the *Seiberg–Witten moduli space* is

$$\mathfrak{M}_{\mathrm{SW}}(\mathbf{p}) := \left\{ [(\Phi, A)] \in \frac{\Gamma(\mathfrak{S}) \times \mathscr{A}_B(Q)}{\mathscr{G}(P)} : \begin{array}{l} (\Phi, A) \text{ satisfies (2.2.1)} \\ \text{with respect to } g \text{ and } B \end{array} \right\}.$$

The *universal Seiberg–Witten moduli space* is

$$\mathfrak{M}_{\mathrm{SW}} := \left\{ (\mathbf{p}, [(\Phi, A)]) \in \mathscr{P} \times \frac{\Gamma(\mathfrak{S}) \times \mathscr{A}(Q)}{\mathscr{G}(P)} : [(\Phi, A)] \in \mathfrak{M}_{\mathrm{SW}}(\mathbf{p}) \right\}.$$

The Seiberg–Witten moduli spaces are endowed with the quotient topology induced from the C^∞–topology on the spaces of connections and sections.

In the upcoming sections of this chapter, we discuss the questions:

1. Is $\mathfrak{M}_{SW}(\mathbf{p})$ a smooth manifold for a generic choice of $\mathbf{p} \in \mathscr{P}$?
2. If $\mathfrak{M}_{SW}(\mathbf{p})$ is smooth, is it orientable?
3. Is $\mathfrak{M}_{SW}(\mathbf{p})$ compact?
4. How does $\mathfrak{M}_{SW}(\mathbf{p})$ depend on the choice of $\mathbf{p} \in \mathscr{P}$?

2.3 DEFORMATION THEORY OF MONOPOLES

The purpose of this section is to describe the structure of $\mathfrak{M}_{SW}$ in a neighborhood of a solution $\mathfrak{c}$ of (2.2.1) for some $\mathbf{p} \in \mathscr{P}$. As we will momentarily see, the deformation theory of (2.2.1) at $(\mathbf{p}, \mathfrak{c})$ is controlled by a differential graded Lie algebra (DGLA). Associated with this DGLA is a self-adjoint elliptic operator $L_{\mathbf{p},\mathfrak{c}}$, which can be understood as a gauge fixed and co-gauge fixed linearization of (2.2.1); a precise definition is given below. These operators equip $\mathfrak{M}_{SW}$ with a real line bundle $\det L$ such that for each $(\mathbf{p}, [\mathfrak{c}]) \in \mathfrak{M}_{SW}$ we have a distinguished isomorphism

$$(\det L)_{(\mathbf{p},[\mathfrak{c}])} \cong \det\ker L_{\mathbf{p},\mathfrak{c}} \otimes (\operatorname{coker} L_{\mathbf{p},\mathfrak{c}})^*.$$

The fact that the operators $L_{\mathbf{p},\mathfrak{c}}$ are Fredholm allows us to construct finite dimensional models of $\mathfrak{M}_{SW}$ by standard methods of Fredholm differential topology. This construction is summarized in the following result.

Proposition 2.3.1. *If $\mathfrak{c}_0$ is a solution of (2.2.1) for $\mathbf{p}_0 \in \mathscr{P}$ and $\mathfrak{c}_0$ is irreducible,[2] then there is a* Kuranishi model *for a neighborhood of $(\mathbf{p}_0, [\mathfrak{c}_0]) \in \mathfrak{M}_{SW}$; that is: there are an open neighborhood of U of $\mathbf{p}_0 \in \mathscr{P}$, finite dimensional vector spaces $I = \ker L_{\mathbf{p}_0,\mathfrak{c}_0}$ and $O = \operatorname{coker} L_{\mathbf{p}_0,\mathfrak{c}_0}$ of the same dimension, an open neighborhood $\mathscr{I}$ of $0 \in I$, a smooth map*

$$\mathrm{ob}\colon U \times \mathscr{I} \to O,$$

an open neighborhood V of $(\mathbf{p}_0, [\mathfrak{c}_0]) \in \mathfrak{M}_{SW}$, and a homeomorphism

$$\mathfrak{x}\colon \mathrm{ob}^{-1}(0) \to V \subset \mathfrak{M}_{SW},$$

which maps $(\mathbf{p}_0, 0)$ to $(\mathbf{p}_0, [\mathfrak{c}_0])$ and commutes with the projections to $\mathscr{P}$. Moreover, for each $(\mathbf{p}, \mathfrak{c}) \in \operatorname{im}\mathfrak{x}$, there is an exact sequence

$$0 \to \ker L_{\mathbf{p},\mathfrak{c}} \to I \xrightarrow{\mathrm{d}_I \mathrm{ob}} O \to \operatorname{coker} L_{\mathbf{p},\mathfrak{c}} \to 0$$

such that the induced maps $\det L_{\mathbf{p},\mathfrak{c}} \to \det(I) \otimes \det(O)^*$ define an isomorphism of line bundles $\det L \cong \mathfrak{x}_*(\det(I) \otimes \det(O)^*)$ on $\operatorname{im}\mathfrak{x} \subset \mathfrak{M}_{SW}$.

Corollary 2.3.2. *If $L_{\mathbf{p}_0,\mathfrak{c}_0}$ is invertible, then the projection $\mathfrak{M}_{SW} \to \mathscr{P}$ is a local homeomorphism at $(\mathbf{p}_0, [\mathfrak{c}_0])$. In particular, $[\mathfrak{c}_0]$ is then an isolated point of $\mathfrak{M}_{SW}(\mathbf{p}_0)$.*

2 In the sense of Definition 2.3.5 given below. There is a natural generalization of Proposition 2.3.1 to the case when $\mathfrak{c}_0$ is reducible. Then the stabilizer of $\mathfrak{c}_0$ in the gauge group acts on U and O and ob can be chosen to be equivariant, cf. [DK90, Section 4.2.5]. In this thesis we focus on irreducible solutions.

In the remaining part of the section we set the deformation theory of monopoles and prove Proposition 2.3.1. We begin with the Seiberg–Witten DGLA. In order to simplify the notation, assume for the remaining part of this section that a parameter $\mathbf{p} \in \mathscr{P}$ has been fixed. Thus, we will suppress it from the notation, writing $L_{\mathfrak{c}}$ instead of $L_{\mathbf{p},\mathfrak{c}}$ and so on. However, keep in mind that most of the operators discussed below depend on $\mathbf{p}$.

Definition 2.3.3. Denote by $L^\bullet$ the graded real vector space given by

$$\begin{aligned} L^0 &:= \Omega^0(M, \mathfrak{g}_P), \\ L^1 &:= \Gamma(\mathfrak{S}) \oplus \Omega^1(M, \mathfrak{g}_P), \\ L^2 &:= \Gamma(\mathfrak{S}) \oplus \Omega^2(M, \mathfrak{g}_P), \quad \text{and} \\ L^3 &:= \Omega^3(M, \mathfrak{g}_P). \end{aligned}$$

Denote by $[\![\cdot,\cdot]\!]\colon L^\bullet \otimes L^\bullet \to L^\bullet$ the graded skew-symmetric bilinear map defined by

$$\begin{aligned} [\![a,b]\!] &:= [a \wedge a] && \text{for } a, b \in \Omega^\bullet(M, \mathfrak{g}_P), \\ [\![\xi,\phi]\!] &:= \rho(\xi)\phi && \text{for } \xi \in \Omega^0(M, \mathfrak{g}_P) \text{ and } \phi \in \Gamma(\mathfrak{S}) \text{ in degree 1 or 2}, \\ [\![a,\phi]\!] &:= -\bar{\gamma}(a)\phi && \text{for } a \in \Omega^1(M, \mathfrak{g}_P) \text{ and } \phi \in \Gamma(\mathfrak{S}) \text{ in degree 1}, \\ [\![\phi,\psi]\!] &:= -2\mu(\phi,\psi) && \text{for } \phi, \psi \in \Gamma(\mathfrak{S}) \text{ in degree 1, and} \\ [\![\phi,\psi]\!] &:= -*\rho^*(\phi\psi^*) && \text{for } \phi \in \Gamma(\mathfrak{S}) \text{ in degree 1 and } \psi \in \Gamma(\mathfrak{S}) \text{ in degree 2}. \end{aligned}$$

Given $\mathfrak{c} = (\Phi, A) \in \Gamma(\mathfrak{S}) \times \mathscr{A}_B(Q)$, define the degree one linear map $\delta^\bullet = \delta^\bullet_{\mathfrak{c}}\colon L^\bullet \to L^{\bullet+1}$ by

$$\begin{aligned} \delta^0_{\mathfrak{c}}(\xi) &:= \begin{pmatrix} -\rho(\xi)\Phi \\ \mathrm{d}_A\xi \end{pmatrix}, \\ \delta^1_{\mathfrak{c}}(\phi, a) &:= \begin{pmatrix} -\not{D}_A\phi - \bar{\gamma}(a)\Phi \\ -2\mu(\Phi,\phi) + \mathrm{d}_A a \end{pmatrix}, \quad \text{and} \\ \delta^2_{\mathfrak{c}}(\psi, b) &:= *\rho^*(\psi\Phi^*) + \mathrm{d}_A b. \end{aligned}$$

Proposition 2.3.4. *The algebraic structures defined in Definition 2.3.3 determine a DGLA which controls the deformation theory of the Seiberg–Witten equation; that is:*

1. *$(L^\bullet, [\![\cdot,\cdot]\!])$ is a graded Lie algebra.*
2. *If $\mathfrak{c} = (\Phi, A)$ is a solution of (2.2.1), then $(L^\bullet, [\![\cdot,\cdot]\!], \delta^\bullet_{\mathfrak{c}})$ is a DGLA.*
3. *Suppose that $\mathfrak{c} = (\Phi, A)$ is a solution of (2.2.1). For every $\hat{\mathfrak{c}} = (\phi, a) \in L^1$, $(\Phi + \phi, A + a)$ solves (2.2.1) if and only if it is a* Maurer–Cartan element, *that is, $\delta_{\mathfrak{c}}\hat{\mathfrak{c}} + \frac{1}{2}[\![\hat{\mathfrak{c}}, \hat{\mathfrak{c}}]\!] = 0$.*

While part (3) is a straightforward calculation, the verification of (1) and (2) is somewhat lengthy. Since it does not play an important role in what follows, we refer to [DW17a, Appendix B].

Definition 2.3.5. Let $\mathfrak{c} \in \Gamma(\mathfrak{S}) \times \mathscr{A}_B(Q)$ be a solution of (2.2.1). We call

$$\Gamma_{\mathfrak{c}} := \{u \in \mathscr{G}(P) : u\mathfrak{c} = \mathfrak{c}\}$$

the group of *automorphisms* of $\mathfrak{c}$. Its Lie algebra is the cohomology group $H^0(L^\bullet, \delta_\mathfrak{c})$; $H^1(L^\bullet, \delta_\mathfrak{c})$ is the space of *infinitesimal deformations*, and $H^2(L^\bullet, \delta_\mathfrak{c})$ the space of *infinitesimal obstructions*.

We say that $\mathfrak{c}$ is *irreducible* if $\Gamma_\mathfrak{c} = 0$, and *unobstructed* if $H^2(L^\bullet, \delta_\mathfrak{c}) = 0$.

Remark 2.3.6. $H^3(L^\bullet, \delta_\mathfrak{c})$ has no immediate interpretation, but it is isomorphic to $H^0(L^\bullet, \delta_\mathfrak{c})$, since the complex $(L^\bullet, \delta_\mathfrak{c})$ is self-dual. The latter also shows that $H^1(L^\bullet, \delta_\mathfrak{c})$ is isomorphic to $H^2(L^\bullet, \delta_\mathfrak{c})$.

The operators

$$\begin{aligned}
\tilde\delta^0_\mathfrak{c} &:= \delta^0_\mathfrak{c} \colon\ \Omega^0(M, \mathfrak{g}_P) \to \Gamma(\mathfrak{S}) \oplus \Omega^1(M, \mathfrak{g}_P),\\
\tilde\delta^1_\mathfrak{c} &:= (\mathrm{id}_\mathfrak{S} \oplus *) \circ \delta^1_\mathfrak{c} \colon\ \Gamma(\mathfrak{S}) \oplus \Omega^1(M, \mathfrak{g}_P) \to \Gamma(\mathfrak{S}) \oplus \Omega^1(M, \mathfrak{g}_P), \quad \text{and}\\
\tilde\delta^2_\mathfrak{c} &:= - * \circ\, \delta^2_\mathfrak{c} \circ (\mathrm{id}_\mathfrak{S} \oplus *) \colon\ \Gamma(\mathfrak{S}) \oplus \Omega^1(M, \mathfrak{g}_P) \to \Omega^0(M, \mathfrak{g}_P)
\end{aligned}$$

satisfy

$$(\tilde\delta^0_\mathfrak{c})^* = \delta^2_\mathfrak{c} \quad \text{and} \quad (\delta^1_\mathfrak{c})^* = \delta^1_\mathfrak{c}.$$

It follows that the operator

$$L_\mathfrak{c} \colon\ \Gamma(\mathfrak{S}) \oplus \Omega^1(M, \mathfrak{g}_P) \oplus \Omega^0(M, \mathfrak{g}_P) \to \Gamma(\mathfrak{S}) \oplus \Omega^1(M, \mathfrak{g}_P) \oplus \Omega^0(M, \mathfrak{g}_P)$$

defined by

$$\begin{aligned}
L_\mathfrak{c} &:= \begin{pmatrix} \tilde\delta^1_\mathfrak{c} & \tilde\delta^0_\mathfrak{c} \\ \tilde\delta^2_\mathfrak{c} & 0 \end{pmatrix}\\
&= \begin{pmatrix} -\not{D}_A & 0 & 0 \\ 0 & *\mathrm{d}_A & \mathrm{d}_A \\ 0 & \mathrm{d}_A^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\bar\gamma(\cdot)\Phi & -\rho(\cdot)\Phi \\ -2 * \mu(\Phi, \cdot) & 0 & 0 \\ -\rho^*(\cdot\, \Phi^*) & 0 & 0 \end{pmatrix}
\end{aligned}$$

is formally self-adjoint and elliptic.

Definition 2.3.7. We call $L_\mathfrak{c}$ the *linearization* of the Seiberg–Witten equation at $\mathfrak{c}$.

If $\mathfrak{c}$ is a solution of (2.2.1), then Hodge theory identifies $H^1(L^\bullet, \delta_\mathfrak{c}) \oplus H^0(L^\bullet, \delta_\mathfrak{c})$ with $\ker L_\mathfrak{c}$ and $H^2(L^\bullet, \delta_\mathfrak{c}) \oplus H^3(L^\bullet, \delta_\mathfrak{c})$ with $\operatorname{coker} L_\mathfrak{c}$. The fact that $(L^\bullet, \delta_\mathfrak{c})$ is self-dual (up to signs) manifests itself as $L_\mathfrak{c}$ being formally self-adjoint. After gauge fixing and co-gauge fixing, we can understand (2.2.1) as an elliptic PDE as follows.

Proposition 2.3.8. *Given*

$$\mathfrak{c}_0 = (\Phi_0, A_0) \in \Gamma(\mathfrak{S}) \times \mathscr{A}_B(Q),$$

define $Q\colon\ \Gamma(\mathfrak{S}) \oplus \Omega^1(M, \mathfrak{g}_P) \oplus \Omega^0(M, \mathfrak{g}_P) \to \Gamma(\mathfrak{S}) \oplus \Omega^1(M, \mathfrak{g}_P) \oplus \Omega^0(M, \mathfrak{g}_P)$ *by*

$$Q(\phi, a, \xi) := \begin{pmatrix} -\bar\gamma(a)\phi \\ \frac{1}{2} * [a \wedge a] - *\mu(\phi) \\ 0 \end{pmatrix},$$

$\mathfrak{e}_{\mathfrak{c}_0} \in \Gamma(\mathfrak{S}) \oplus \Omega^1(M, \mathfrak{g}_P) \oplus \Omega^0(M, \mathfrak{g}_P)$ *by*

$$\mathfrak{e}_{\mathfrak{c}_0} := \begin{pmatrix} -\not{D}_{A_0}\Phi_0 \\ *\varpi F_{A_0} - *\mu(\Phi_0) \\ 0 \end{pmatrix},$$

and set

$$\mathfrak{sw}_{\mathfrak{c}_0}(\hat{\mathfrak{c}}) := L_{\mathfrak{c}_0}\hat{\mathfrak{c}} + Q_{\mathfrak{c}_0}(\hat{\mathfrak{c}}) + \mathfrak{e}_{\mathfrak{c}_0}.$$

There is a constant $\sigma > 0$ depending on $\mathfrak{c}_0$ such that, for every $\hat{\mathfrak{c}} = (\phi, a, \xi) \in \Gamma(\mathfrak{S}) \oplus \Omega^1(M, \mathfrak{g}_P) \oplus \Omega^0(M, \mathfrak{g}_P)$ satisfying $\|(\phi, a)\|_{L^\infty} < \sigma$, the equation

$$\mathfrak{sw}_{\mathfrak{c}_0}(\hat{\mathfrak{c}}) = 0$$

holds if and only if $\mathfrak{c}_0 + (\phi, a)$ satisfies (2.2.1) and the gauge fixing condition

$$\mathrm{d}_{A_0}^* a - \rho^*(\phi\Phi_0^*) = 0 \tag{2.3.1}$$

as well as the co-gauge fixing condition

$$\mathrm{d}_{A_0}\xi = 0 \quad \textit{and} \quad \rho(\xi)\Phi_0 = 0;$$

moreover, if $\mathfrak{c}_0$ is infinitesimally irreducible (that is: $H^0(L^\bullet, \delta_{\mathfrak{c}_0}) = 0$), then $\xi = 0$.

The proof requires a number of useful identities for μ which are collected in the next proposition. We refer to [DW17a, Proposition B.4] for a straightforward but rather tedious derivation of these identities.

Proposition 2.3.9. *For all $A \in \mathscr{A}(Q)$ and $\phi, \psi \in \Gamma(\mathfrak{S})$ we have*

$$\mathrm{d}_A\mu(\phi,\psi) = - * \frac{1}{2}\rho^*\left((\not{D}_A\phi)\psi^* + (\not{D}_A\psi)\phi^*\right) \tag{2.3.2}$$

and

$$\begin{aligned} \mathrm{d}_A^*\mu(\phi,\psi) = {} & *\mu(\not{D}_A\phi, \psi) + *\mu(\not{D}_A\psi, \phi) \\ & - \frac{1}{2}\rho^*\left((\nabla_A\phi)\psi^*\right) - \frac{1}{2}\rho^*\left((\nabla_A\psi)\phi^*\right). \end{aligned} \tag{2.3.3}$$

Proof of Proposition 2.3.8. Setting $\Phi := \Phi_0 + \phi$ and $A := A_0 + a$, the equation $\mathfrak{sw}_{\mathfrak{c}_0}(\hat{\mathfrak{c}}) = 0$ amounts to

$$\begin{aligned} \not{D}_A\Phi + \rho(\xi)\Phi_0 &= 0, \\ \varpi F_A + *\mathrm{d}_{A_0}\xi &= \mu(\Phi), \quad \text{and} \\ \mathrm{d}_{A_0}^* a - \rho^*(\phi\Phi_0^*) &= 0. \end{aligned}$$

Since

$$\mathrm{d}_A\mu(\Phi) = - * \rho^*\left((\not{D}_A\Phi)\Phi^*\right)$$

by (2.3.2), applying d_A to the second equation above and using the first equation we obtain

$$\mathrm{d}_{A_0}^*\mathrm{d}_{A_0}\xi + \rho^*\left((\rho(\xi)\Phi_0)\Phi_0^*\right) - *[a \wedge *\mathrm{d}_{A_0}\xi] + \rho^*\left((\rho(\xi)\Phi_0)\phi^*\right) = 0.$$

Taking the L^2 inner product with ξ_0, the component of ξ in the L^2 orthogonal complement of $\ker \delta_{\mathfrak{c}_0}$ and integrating by parts yields that

$$\|\mathrm{d}_{A_0}\xi\|_{L^2}^2 + \|\rho(\xi)\Phi_0\|_{L^2}^2 = \langle *[a \wedge *\mathrm{d}_{A_0}\xi], \xi_0\rangle_{L^2} - \langle \rho(\xi)\Phi_0, \rho(\xi_0)\phi\rangle_{L^2}.$$

The right-hand side can be bounded by a constant $c > 0$ (depending on $\mathfrak{c}_0$) times

$$\|(a,\phi)\|_{L^\infty}\left(\|\mathrm{d}_{A_0}^*\xi\|_{L^2}^2 + \|\rho(\xi)\Phi_0\|_{L^2}^2\right).$$

Therefore, if $\|(a,\phi)\|_{L^\infty} < \sigma := 1/c$, then

$$\mathrm{d}_{A_0}\xi = 0 \quad \text{and} \quad \rho(\xi)\Phi_0 = 0.$$

It follows that $\hat{\mathfrak{c}} + (\phi, a)$ satisfies (2.2.1).

Since $\xi \in H^0(L^\bullet, \delta_{\mathfrak{c}_0})$, it vanishes if $\mathfrak{c}_0$ infinitesimally irreducible. □

The following standard observation shows that imposing the gauge fixing condition (2.3.1) is mostly harmless, as long as we are only interested in small variations $\hat{\mathfrak{c}}$; c.f. [DK90, Proposition 4.2.9].

Remark 2.3.10. In what follows we denote by $W^{k,p}\Gamma(\mathfrak{S})$ the space of sections of $\mathfrak{S}$ of Sobolev class $W^{k,p}$. We use similar notations for spaces of connections, gauge transformations, and differential forms.

Proposition 2.3.11. *Fix $k \in \mathbf{N}$ and $p \in (1,\infty)$ with $(k+1)p > 3$. Given*

$$\mathfrak{c}_0 = (\Phi_0, A_0) \in W^{k+1,p}\Gamma(\mathfrak{S}) \times W^{k+2,p}\mathscr{A}_B(Q),$$

there is a constant $\sigma > 0$ such that if we set

$$\mathfrak{U}_{\mathfrak{c}_0,\sigma} := \left\{\hat{\mathfrak{c}} \in B_\sigma(0) \subset W^{k+1,p}\Gamma(\mathfrak{S}) \times W^{k+2,p}\Omega^1(M, \mathfrak{g}_P) : \mathrm{d}_{A_0}^* a - \rho^*(\phi\Phi_0^*) = 0\right\},$$

then the map

$$\mathfrak{U}_{\mathfrak{c}_0,\sigma}/\Gamma_{\mathfrak{c}_0} \ni [\hat{\mathfrak{c}}] \mapsto [\mathfrak{c}_0 + \hat{\mathfrak{c}}] \in \frac{W^{k+1,p}\Gamma(\mathfrak{S}) \times W^{k+2,p}\mathscr{A}_B(Q)}{W^{k+3,p}\mathscr{G}(P)}$$

is a homeomorphism onto its image; moreover, $\Gamma_{\mathfrak{c}_0+\hat{\mathfrak{c}}}$ is the stabilizer of $\hat{\mathfrak{c}}$ in $\Gamma_{\mathfrak{c}}$.

For $\hat{\mathfrak{c}} = (\phi, a, \xi)$ and $(\Phi, A) = \mathfrak{c} = \mathfrak{c}_0 + (\phi, a)$, we have

$$(\mathrm{d}\mathfrak{sw}_{\mathfrak{c}_0})_{\hat{\mathfrak{c}}} = \begin{pmatrix} -\not{D}_A & 0 & 0 \\ 0 & *\mathrm{d}_A & \mathrm{d}_{A_0} \\ 0 & \mathrm{d}_{A_0}^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\bar{\gamma}(\cdot)\Phi & -\rho(\cdot)\Phi_0 \\ -2*\mu(\Phi,\cdot) & 0 & 0 \\ -\rho^*(\cdot\,\Phi_0^*) & 0 & 0 \end{pmatrix}.$$

In particular, $(\mathrm{d}\mathfrak{sw}_{\mathfrak{c}_0})_0$ agrees with $L_{\mathfrak{c}_0}$. The following proposition explains the relation between $(\mathrm{d}\mathfrak{sw}_{\mathfrak{c}_0})_{\hat{\mathfrak{c}}}$ and $L_{\mathfrak{c}}$ for $\mathfrak{c} = (\Phi, A, 0) + \hat{\mathfrak{c}}$.

Proposition 2.3.12. *In the situation of Proposition 2.3.11, if $\hat{\mathfrak{c}} \in \mathfrak{U}_{\mathfrak{c}_0,\sigma}$ and $\mathfrak{c} = \mathfrak{c}_0 + \hat{\mathfrak{c}}$, then there is a $\tau > 0$ and a smooth map $\phi_{\mathfrak{c}_0,\mathfrak{c}} \colon B_\tau(\mathfrak{c}) \to B_\sigma(0)$ which maps $\mathfrak{U}_{\mathfrak{c},\tau}$ to $\mathfrak{U}_{\mathfrak{c}_0,\sigma}$, commutes with the projection to $\left(W^{k+1,p}\Gamma(\mathfrak{S}) \times W^{k+2,p}\mathscr{A}_B(Q)\right)/\left(W^{k+3,p}\mathscr{G}(P)\right)$, and satisfies*

$$(\mathrm{d}\phi)_{\mathfrak{c}}^{-1}(\mathrm{d}\mathfrak{sw}_{\mathfrak{c}_0})_{\hat{\mathfrak{c}}}(\mathrm{d}\phi)_{\mathfrak{c}} = (\mathrm{d}\mathfrak{sw}_{\mathfrak{c}})_0 = L_{\mathfrak{c}}.$$

Finally, we prove Proposition 2.3.1. The method of proof is standard, c.f. [DK90, Section 4.2]. Fix $k \in \mathbf{N}$ and $p \in (1, \infty)$ with $(k+1)p > 3$. Given $\mathbf{p} = (g, B) \in \mathscr{P}$, set

$$\mathfrak{M}^{k,p}_{\mathrm{SW}}(\mathbf{p}) := \left\{ [(\Phi, A)] \in \frac{W^{k+1,p}\Gamma(\mathfrak{S}) \times W^{k+2,p}\mathscr{A}_B(Q)}{W^{k+3,p}\mathscr{G}(P)} : \begin{array}{l}(\Phi, A) \text{ satisfies (2.2.1)} \\ \text{with respect to } g \text{ and } B\end{array} \right\},$$

and define $\mathfrak{M}^{k,p}_{\mathrm{SW}}$ accordingly. It is a consequence of elliptic regularity for $L_{\mathfrak{c}}$ and Proposition 2.3.11, that the inclusion $\mathfrak{M}_{\mathrm{SW}} \subset \mathfrak{M}^{k,p}_{\mathrm{SW}}$ is a homeomorphism. This together with Proposition 2.3.8 and Proposition 2.3.11 implies that if $(\mathbf{p}_0, [\hat{\mathfrak{c}}_0]) \in \mathfrak{M}_{\mathrm{SW}}$ is irreducible, then there is a constant $\sigma > 0$ and an open neighborhood U of $\mathbf{p} \in \mathscr{P}$ such that if $B_\sigma(0)$ denotes the open ball of radius σ centered at 0 in $W^{k+1,p}\Gamma(\mathfrak{S}) \oplus W^{k+2,p}\Omega^1(M, \mathfrak{g}_P) \oplus W^{k+2,p}\Omega^0(M, \mathfrak{g}_P)$, then

$$\{(\mathbf{p}, \hat{\mathfrak{c}}) \in U \times B_\sigma(0) : \mathfrak{sw}_{\mathbf{p},\mathfrak{c}_0}(\hat{\mathfrak{c}}) = 0\} \ni (\mathbf{p}, [(\phi, a, \xi)]) \mapsto (\mathbf{p}, [\mathfrak{c} + (\phi, a)]) \in \mathfrak{M}_{\mathrm{SW}}$$

is a homeomorphism onto its image. Here we use subscripts to denote the dependence of $L_{\mathfrak{c}_0}$, Q, $\mathfrak{e}_{\mathfrak{c}_0}$, and $\mathfrak{sw}_{\mathfrak{c}_0}$ on the parameter $\mathbf{p} \in \mathscr{P}$. The proof of Proposition 2.3.1 is completed by applying the following result to $\mathfrak{sw}_{\mathbf{p},\mathfrak{c}_0}$ with $I = \ker L_{\mathbf{p}_0,\mathfrak{c}_0}$ and $O = \operatorname{coker} L_{\mathbf{p}_0,\mathfrak{c}_0}$.

Lemma 2.3.13. *Let X and Y be Banach spaces, let $U \subset X$ be a neighborhood of $0 \in X$, let P be a Banach manifold, and let $F\colon P \times U \to Y$ be a smooth map of the form*

$$F(p, x) = L(p, x) + Q(p, x) + \mathfrak{e}(p)$$

such that:

1. *L is smooth, for each $p \in P$, $L_p := L(p, \cdot)\colon X \to Y$ is a Fredholm operator, and we have $\sup_{p \in P} \|L_p\|_{\mathcal{L}(X,Y)} < \infty$,*
2. *Q is smooth and there exists a $c_Q > 0$ such that, for all $x_1, x_2 \in X$ and all $p \in P$, we have*
$$\|Q(x_1, p) - Q(x_2, p)\|_Y \leqslant c_Q \left(\|x_1\|_X + \|x_2\|_X\right) \|x_1 - x_2\|_X, \tag{2.3.4}$$
and
3. *$\mathfrak{e}\colon P \to Y$ is smooth and there is a constant $c_{\mathfrak{e}}$ such that $\|\mathfrak{e}\|_Y \leqslant c_{\mathfrak{e}}$.*

Let $I \subset X$ be a finite dimensional subspace and let $\pi\colon X \to I$ be a projection onto I. Let $O \subset Y$ be a finite dimensional subspace, let $\Pi\colon Y \to O$ be a projection onto O, and denote by $\iota\colon O \to Y$ the inclusion. Suppose that, for all $p \in P$, the operator $\bar{L}_p\colon O \oplus X \to I \oplus Y$ defined by

$$\bar{L}_p := \begin{pmatrix} 0 & \pi \\ \iota & L_p \end{pmatrix}$$

is invertible, and suppose that $c_R := \sup_{p \in P} \|\bar{L}_p^{-1}\|_{\mathcal{L}(Y,X)} < \infty$.

If $c_{\mathfrak{e}} \ll_{c_R, c_Q} 1$, then there is an open neighborhood $\mathscr{I}$ of $0 \in I$, an open subset $V \subset P \times U$ containing $P \times \{0\}$, and a smooth map

$$x\colon P \times \mathscr{I} \to X$$

such that, for each $(p, x_0) \in \mathscr{I} \times P$, $(p, x(p, x_0))$ *is the unique solution* $(p, x) \in V$ *of*

$$(\mathrm{id}_Y - \Pi)F(p, x) = 0 \quad \textit{and} \quad \pi x = x_0. \tag{2.3.5}$$

In particular, if we define $\mathrm{ob}\colon P \times \mathscr{I} \to O$ *by*

$$\mathrm{ob}(p, x_0) := \Pi F(p, x(p, x_0)),$$

then the map $\mathrm{ob}^{-1}(0) \to F^{-1}(0) \cap V$ *defined by*

$$(p, x_0) \mapsto (p, x(p, x_0))$$

is a homeomorphism. Moreover, for every $(p, x_0) \in P \times \mathscr{I}$ *and* $x = x(p, x_0)$*, we have an exact sequence*

$$0 \to \ker \partial_x F(p, x) \to I \xrightarrow{\partial_{x_0}\mathrm{ob}(p, x_0)} O \to \operatorname{coker} \partial_x F(p, x) \to 0;$$

which induces an isomorphism $\det \partial_x F \cong \det I \otimes (\det O)^*$.

This is result is essentially a summary of the discussion in [GW13, Section 5]; see also [DK90, Proposition 4.2.4] or [DW17a, Lemma 3.1].

2.4 ORIENTATIONS

For the purpose of counting solutions to (2.2.1) orientations play an important role. Recall that if the moduli space $\mathfrak{M}_{\mathrm{SW}}$ is smooth, then it can be oriented using a trivialization of the determinant line bundle $\det L$, if such a trivialization exists. The following is a useful criterion to check whether $\det L$ can be trivialized over the configuration space $\mathscr{P} \times \Gamma(\mathfrak{S}) \times \mathscr{A}(Q)/\mathscr{G}(P)$ and, consequently, over $\mathfrak{M}_{\mathrm{SW}}$.

Proposition 2.4.1. *Suppose that algebraic data as in Definition 2.1.7 and compatible geometric data as in Definition 2.2.1 have been fixed. Let* $\rho_G\colon G \to \mathrm{Sp}(S)$ *be the restriction of the quaternionic representation* $\rho\colon H \to \mathrm{Sp}(S)$ *to* $G \triangleleft H$*. Denote by* $c_2 \in B\mathrm{Sp}(S)$ *the universal second Chern class. If* $(B\rho_G)^* c_2 \in H^4(BG, \mathbf{Z})$ *can be written as*

$$(B\rho_G)^* c_2 = 2x + \alpha_1 y_1^2 + \cdots + \alpha_k y_k^2 \tag{2.4.1}$$

with $x \in H^4(BG, \mathbf{Z})$, $y_1, \ldots, y_k \in H^2(BG, \mathbf{Z})$*, and* $\alpha_1, \ldots, \alpha_k \in \mathbf{Z}$*, then*

$$\det L \to \mathscr{P} \times \frac{\Gamma(\mathfrak{S}) \times \mathscr{A}(Q)}{\mathscr{G}(P)}$$

is trivial.

Proof. The parameter space $\mathscr{P}$ is contractible; hence, it is enough to fix an element $\mathbf{p} \in \mathscr{P}$ and prove that $\det L$ is trivial over the second factor. We need to show that if $(\mathfrak{c}_t)_{t\in[0,1]}$ is a path in $\Gamma(\mathfrak{S}) \times \mathscr{A}_B(Q)$ and $u \in \mathscr{G}(P)$ is such that $u\mathfrak{c}_1 = \mathfrak{c}_0$, then the spectral flow of $(L_{\mathfrak{c}_t})_{t\in[0,1]}$ is even. The mapping torus of $u\colon Q \to Q$ is a principal H–bundle $\mathbf{Q}$ over $S^1 \times M$, and the path $(\mathfrak{c}_t)_{t\in[0,1]}$ induces a connection $\mathbf{A}$ on $\mathbf{Q}$. Over $S^1 \times M$ we also have an adjoint bundle $\mathfrak{g}_{\mathbf{P}}$ and the spinor bundles $\mathfrak{S}^+$ and $\mathfrak{S}^-$ associated with $\mathbf{Q}$ via the quaternionic representation $\rho\colon H \to \mathrm{Sp}(S)$.

According to Atiyah–Singer–Patodi, the spectral flow of $(L_{\mathfrak{c}_t})_{t\in[0,1]}$ is the index of the operator $\mathbf{L} = \partial_t - L_{\mathfrak{c}_t}$ which can be identified with an operator

$$\mathbf{L}\colon \Gamma(\mathfrak{S}^+) \oplus \Omega^1(S^1 \times M, \mathfrak{g}_\mathbf{P}) \to \Gamma(\mathfrak{S}^-) \oplus \Omega^+(S^1 \times M, \mathfrak{g}_\mathbf{P}) \oplus \Omega^0(S^1 \times M, \mathfrak{g}_\mathbf{P}).$$

In our case, $\mathbf{L}$ is homotopic through Fredholm operators to the sum of the Dirac operator $\not{D}^+_\mathbf{A}\colon \Gamma(\mathfrak{S}^+) \to \Gamma(\mathfrak{S}^-)$ and the Atiyah–Hitchin–Singer operator $\mathrm{d}^+_\mathbf{A} \oplus \mathrm{d}^*_\mathbf{A}$ for $\mathfrak{g}_\mathbf{P}$. The index of the Atiyah–Hitchin–Singer operator is $-2p_1(\mathfrak{g}_\mathbf{P})$ and thus even. To compute the index of the Dirac operator, observe that the vector bundle $\mathbf{V} := \mathbf{Q} \times_\rho S$ inherits from S the structure of a left-module over $\mathbf{H}$ and that

$$\mathfrak{S}^\pm = \not{S}^\pm \otimes_\mathbf{H} \mathbf{V},$$

where $\not{S}^\pm$ are the usual spinor bundles of $S^1 \times M$ with the spin structure induced from that on M and we use the structure of $\not{S}^\pm$ as a right-modules over $\mathbf{H}$. $\mathfrak{S}^\pm$ is a real vector bundle: it is a real form of $\not{S}^\pm \otimes_\mathbf{C} \mathbf{V}$. Therefore, the complexification of $\not{D}^+_\mathbf{A}$ is the standard complex Dirac operator on $\not{S}^\pm$ twisted by $\mathbf{V}$. By the Atiyah–Singer Index Theorem,

$$\begin{aligned}\operatorname{index} \not{D}^+_\mathbf{A} &= \int_{S^1\times M} \hat{A}(S^1 \times M)\mathrm{ch}(\mathbf{V}) \\ &= \int_{S^1\times M} \mathrm{ch}_2(\mathbf{V}) = -\int_{S^1\times M} c_2(\mathbf{V}).\end{aligned}$$

The classifying map $f_\mathbf{V}\colon S^1 \times M \to B\mathrm{Sp}(S)$ of $\mathbf{V}$ is related to the classifying map $f_\mathbf{Q}\colon S^1 \times M \to BG$ of $\mathbf{Q}$ through

$$f_\mathbf{V} = B\rho_G \circ f_\mathbf{Q},$$

and

$$c_2(\mathbf{V}) = f^*_\mathbf{V} c_2 = f^*_\mathbf{Q}(B\rho_G)^* c_2.$$

Since the intersection form of $S^1 \times M$ is even, the hypothesis implies that the right-hand side of the above index formula is even. □

Remark 2.4.2. If G is simply–connected, then the condition (2.4.1) is satisfied if and only if the image of

$$(\rho_G)_*\colon \pi_3(G) \to \pi_3(\mathrm{Sp}(S)) = \mathbf{Z}$$

is generated by an even integer. To see this, observe that BG is 3–connected; hence, by the Hurewicz theorem, $H_4(BG, \mathbf{Z}) = \pi_4(BG) \cong \pi_3(G)$ and $H_i(BG, \mathbf{Z}) = 0$ for $i = 1, 2, 3$. The same is true for $\mathrm{Sp}(S)$, and we have a commutative diagram

$$\begin{array}{ccc} H_4(BG, \mathbf{Z}) & \xrightarrow{(B\rho_G)_*} & H_4(B\mathrm{Sp}(S), \mathbf{Z}) \\ \downarrow\cong & & \downarrow\cong \\ \pi_3(G) & \xrightarrow{(\rho_G)_*} & \pi_3(\mathrm{Sp}(S)). \end{array}$$

The group $H_4(BG, \mathbf{Z})$ is freely generated by some elements $x_1, \dots, x_k$. Let $x^1, \dots, x^k$ be the dual basis of $H^4(BG, \mathbf{Z}) = \mathrm{Hom}(H_4(BG, \mathbf{Z}), \mathbf{Z})$. Likewise, $H_4(B\mathrm{Sp}(S), \mathbf{Z})$ is freely generated by the unique element z satisfying $\langle c_2, z\rangle = 1$. We have

$$(B\rho_G)^* c_2 = \sum_{i=1}^k \langle (B\rho_G)^* c_2, x_i\rangle x^i \tag{2.4.2}$$

and

$$\langle (B\rho_G)^* c_2, x_i \rangle = \langle c_2, (B\rho_G)_* x_i \rangle.$$

Therefore, the coefficients in the sum (2.4.2) are all even if and only if the image of $(B\rho_G)_*$ is generated by $2mz$ for some $m \in \mathbf{Z}$.

Example 2.4.3. The hypothesis of Proposition 2.4.1 holds when $S = \mathbf{H} \otimes_{\mathbf{C}} W$ for some complex Hermitian vector space W of dimension n and ρ_G is induced from a unitary representation $G \to \mathrm{U}(W)$; as is the case for the representations leading to the classical Seiberg–Witten and $\mathrm{U}(n)$–monopole equations, see Example 2.2.9 and Example 2.2.10. To see that $(B\rho_G)^* c_2$ is of the desired form, note that if E is a rank n Hermitian vector bundle, then the corresponding quaternionic Hermitian bundle obtained via the inclusion $\mathrm{U}(n) \to \mathrm{Sp}(n)$ is $\mathbf{H} \otimes_{\mathbf{C}} E = E \oplus \bar{E}$ and

$$c_2(\mathbf{H} \otimes_{\mathbf{C}} E) = c_2(E \oplus \bar{E}) = 2c_2(E) - c_1(E)^2.$$

Example 2.4.4. The hypothesis of Proposition 2.4.1 is also satisfied when $S = \mathbf{H} \otimes_{\mathbf{R}} W$ for a real Euclidean vector space W, and ρ_G is induced from an orthogonal representation $G \to \mathrm{SO}(W)$; as is the case for the equation for flat $G^{\mathbf{C}}$–connections, see Example 2.2.11. To see that $(B\rho_G)^* c_2$ is of the desired form, note that if E is a Euclidean vector bundle of rank n, then the associated quaternionic Hermitian vector bundle is $\mathbf{H} \otimes_{\mathbf{R}} E$ and

$$c_2(\mathbf{H} \otimes_{\mathbf{R}} E) = -2p_1(E).$$

If two quaternionic representations satisfy the hypothesis of Proposition 2.4.1, then so does their direct sum. Therefore, the previous two examples together show that $\det L$ is trivial for the ADHM Seiberg–Witten equation, which will be discussed in Section 2.9.

Example 2.4.5. In general, $\det L$ need not be orientable. If $S = \mathbf{H}$ and $G = H = \mathrm{Sp}(1)$ acts on S by right multiplication, then it is easy to see that the gauge transformation of the trivial bundle $Q = S^3 \times \mathrm{SU}(2)$ induced by $S^3 \cong \mathrm{SU}(2)$ gives rise to an odd spectral flow.

2.5 FUETER SECTIONS AND THE HAYDYS CORRESPONDENCE

Unless $\mu^{-1}(0) = \{0\}$, the projection map $\mathfrak{M}_{\mathrm{SW}} \to \mathscr{P}$ is not expected to be proper. In particular, the moduli space $\mathfrak{M}_{\mathrm{SW}}(\mathbf{p})$ might fail to be compact for some $\mathbf{p} \in \mathscr{P}$. This potential non-compactness phenomenon is related to the lack of a priori bounds on Φ for (Φ, A) a solution of (2.2.1). With this in mind, we blow-up the equation (2.2.1); cf. [KM07, Section 2.5] and [HW15, Equation (1.4)].

Definition 2.5.1. The *blown-up Seiberg–Witten equation* is the following differential equation for $(\varepsilon, \Phi, A) \in [0, \infty) \times \Gamma(\mathfrak{S}) \times \mathscr{A}_B(Q)$:

$$\begin{aligned} &\not{D}_A \Phi = 0, \\ &\varepsilon^2 \varpi F_A = \mu(\Phi), \quad \text{and} \\ &\|\Phi\|_{L^2} = 1. \end{aligned} \tag{2.5.1}$$

The non-compactness of $\mathfrak{M}_{\mathrm{SW}}(\mathbf{p})$ can be understood by analyzing sequences of solutions of the above equation for which ε tends to zero. Precisely understanding the limits of such sequences is one of the central challenges in this subject. From work of Taubes [Tau13b] and Haydys and Walpuski [HW15] we expect that the compactification of $\mathfrak{M}_{\mathrm{SW}}(\mathbf{p})$ will contain solutions of (2.5.1) with $\varepsilon = 0$ which are possibly singular along a one-dimensional subset of M. This work leads to the following compactness conjecture for Seiberg–Witten equations. Set

$$\mathfrak{S}^{\mathrm{reg}} := (\mathfrak{s} \times Q) \times_{\mathrm{Sp}(1)\times H} S^{\mathrm{reg}},$$

where $S^{\mathrm{reg}} \subset S$ is the set of regular elements introduced in Definition 2.1.4.

Conjecture 2.5.2. *Let $(\varepsilon_i, \Phi_i, A_i)$ be a sequence of solutions to (2.5.1) such that $\varepsilon_i \to 0$. Then there exist*

1. *a closed, 1–rectifiable set $Z \subset M$ of finite 1–dimensional Hausdorff measure,*
2. *a pair $(\Phi_\infty, A_\infty) \in \Gamma(M\backslash Z, \mathfrak{S}^{\mathrm{reg}}) \times \mathscr{A}_B(M\backslash Z, Q)$*

such that, after passing to a subsequence, $(\Phi_i, A_i) \to (\Phi_\infty, A_\infty)$ modulo gauge in C^∞_{loc} on $M\backslash Z$, and (Φ_∞, A_∞) satisfies (2.5.1) with $\varepsilon = 0$ on $M\backslash Z$.

In general, Φ_∞ cannot be extended to a continuous section of $\mathfrak{S}$ over all of M. However, one expects that the induced section of the bundle of hyperkähler quotients $\mu^{-1}(0) \cap \mathfrak{S}^{\mathrm{reg}}/G$ extends to a continuous section of $\mu^{-1}(0)/G$, which is defined over all of M and takes values in the singular strata of the hyperkähler quotient along $Z \subset M$, cf. Remark 2.1.6.

In Section 2.9 we will discuss some cases for which a version of Conjecture 2.5.2 has been proved. The appearance of a singular set Z poses significant problems in developing compactness theory for $\mathfrak{M}_{\mathrm{SW}}(\mathbf{p})$. However, for the remaining part of this section, let us ignore this difficulty and focus on understanding solutions (Φ, A) to (2.5.1) with $\varepsilon = 0$, assuming that Z is empty, that is:

1. Φ and A are defined over the entire 3–manifold M,
2. Φ takes values in $\mathfrak{S}^{\mathrm{reg}}$.

Definition 2.5.3. The *partially compactified Seiberg–Witten moduli space* is

$$\overline{\mathfrak{M}}_{\mathrm{SW}}(g,B) := \left\{ (\varepsilon, [(\Phi, A)]) \in [0,\infty) \times \frac{\Gamma(\mathfrak{S}) \times \mathscr{A}_B(Q)}{\mathscr{G}(P)} : \begin{array}{l} (\varepsilon, \Phi, A) \text{ satisfies (2.5.1)} \\ \text{with respect to } g \text{ and } B; \\ \text{if } \varepsilon = 0, \text{ then } \Phi \in \Gamma(\mathfrak{S}^{\mathrm{reg}}) \end{array} \right\}.$$

Likewise, the *universal partially compactified Seiberg–Witten moduli space* is

$$\overline{\mathfrak{M}}_{\mathrm{SW}} := \left\{ (\mathbf{p}, \varepsilon, [(\Phi, A)]) \in \mathscr{P} \times [0,\infty) \times \frac{\Gamma(\mathfrak{S}) \times \mathscr{A}(Q)}{\mathscr{G}(P)} : (\varepsilon, [(\Phi, A)]) \in \overline{\mathfrak{M}}_{\mathrm{SW}}(\mathbf{p}) \right\}.$$

The partially compactified Seiberg–Witten moduli spaces are also naturally topological spaces. The formal boundary of $\overline{\mathfrak{M}}_{\mathrm{SW}}$ is

$$\partial\mathfrak{M}_{\mathrm{SW}} := \{(\mathbf{p}, 0, [(\Phi, A)]) \in \overline{\mathfrak{M}}_{\mathrm{SW}}\},$$

and the map

$$\overline{\mathfrak{M}}_{\mathrm{SW}}\backslash\partial\mathfrak{M}_{\mathrm{SW}} \to \mathfrak{M}_{\mathrm{SW}}, \quad (\mathbf{p}, \varepsilon, [(\Phi, A)]) \mapsto (\mathbf{p}, [(\varepsilon^{-1}\Phi, A)])$$

is a homeomorphism. This justifies the term "partially compactified".

Remark 2.5.4. The space $\overline{\mathfrak{M}}_{\mathrm{SW}}(g, B)$ need not be compact, since the actual compactification should include also solutions of the limiting equation with a possibly non-empty singular set Z. In fact, even the space $\partial\mathfrak{M}_{\mathrm{SW}}$ need not be compact [Wal15]. One should think of $\overline{\mathfrak{M}}_{\mathrm{SW}}(g, B)$ as a subset of the actual compactification, which is yet to be defined. We return to this issue in Chapter 4.

For $\varepsilon = 0$, equation (2.5.1) appears to be a degenerate partial differential equation. However, since $\Phi \in \Gamma(\mathfrak{S}^{\mathrm{reg}})$, this equation can be understood as an elliptic PDE as follows.

Definition 2.5.5. The *bundle of hyperkähler quotients* $\pi\colon \mathfrak{X} \to M$ is

$$\mathfrak{X} := (\mathfrak{s} \times R) \times_{\mathrm{Sp}(1)\times K} X.$$

Its *vertical tangent bundle* is

$$V\mathfrak{X} := (\mathfrak{s} \times R) \times_{\mathrm{Sp}(1)\times K} TX,$$

and $\gamma\colon\ \mathrm{Im}\,\mathbf{H} \to \mathrm{End}(S)$ induces a *Clifford multiplication* $\gamma\colon\ \pi^*TM \to \mathrm{End}(V\mathfrak{X})$.

Definition 2.5.6. Using $B \in \mathscr{A}(R)$ we can assign to each $s \in \Gamma(\mathfrak{X})$ its covariant derivative $\nabla_B s \in \Omega^1(M, s^*V\mathfrak{X})$. A section $s \in \Gamma(\mathfrak{X})$ is called a *Fueter section* if it satisfies the *Fueter equation*

$$\mathfrak{F}(s) = \mathfrak{F}_B(s) := \gamma(\nabla_B s) = 0 \in \Gamma(s^*V\mathfrak{X}). \tag{2.5.2}$$

The map $s \mapsto \mathfrak{F}(s)$ is called the *Fueter operator*.[3]

The linearized Fueter operator $(\mathrm{d}\mathfrak{F})_s\colon\ \Gamma(s^*V\mathfrak{X}) \to \Gamma(s^*V\mathfrak{X})$ is a formally self-adjoint elliptic differential operator of order one. In particular, it is Fredholm of index zero. However, the space of solutions to $\mathfrak{F}(s) = 0$, if non-empty, is never zero-dimensional. The reason is that the hyperkähler quotient $X = S^{\mathrm{reg}} /\!/\!/ G$ carries a free $\mathbf{R}^+$–action inherited from the vector space structure on S. This induces a fiber-preserving action of $\mathbf{R}^+$ on $\mathfrak{X}$. One easily verifies that, for $\lambda \in \mathbf{R}^+$ and $s \in \Gamma(\mathfrak{X})$,

$$\mathfrak{F}(\lambda s) = \lambda \mathfrak{F}(s). \tag{2.5.3}$$

As a result, $\mathbf{R}^+$ acts freely on the space of solutions to (2.5.2) which shows that Fueter sections come in one-parameter families. At the infinitesimal level, this shows that every Fueter section is obstructed.

Definition 2.5.7. The *radial vector field* $\hat{v} \in \Gamma(\mathfrak{X}, V\mathfrak{X})$ is the vector field generating the $\mathbf{R}^+$–action on $\mathfrak{X}$.

Differentiating (2.5.3) shows that if s is a Fueter section, then $\hat{v} \circ s \in \Gamma(s^*V\mathfrak{X})$ is a non-zero element of $\ker(\mathrm{d}\mathfrak{F})_s$.

Note that every section of $\Phi \in \Gamma(\mathfrak{S}^{\mathrm{reg}})$ satisfying $\mu(\Phi) = 0$ gives rise to a section s of $\mathfrak{X}$, via the projection $p\colon\ \mu^{-1}(0) \to X = \mu^{-1}(0)/G$. A theorem of Haydys [Hay12, Section 4.1] asserts that the map $\Phi \mapsto s$ induces a homeomorphism between the moduli space of solutions to (2.5.1) with $\varepsilon = 0$ and the space of sections of $\mathfrak{X}$ satisfying the Fueter equation. The next two propositions explain in detail Haydys' theorem and its proof. In what follows, we use the notation introduced in Proposition 2.1.5.

3 In the following, we will suppress the subscript B from the notation.

Proposition 2.5.8. *Given a set of geometric data as in Definition 2.2.1, set*

$$X := S^{\mathrm{reg}} /\!/\!/ G \quad \text{and} \quad \mathfrak{X} := (\mathfrak{s} \times R) \times_{\mathrm{Sp}(1)\times K} X.$$

Denote by $p\colon S^{\mathrm{reg}} \cap \mu^{-1}(0) \to X$ the canonical projection.

1. *If $s \in \Gamma(\mathfrak{X})$, then there exist a principal H–bundle Q together with an isomorphism $Q \times_H K \cong R$ and a section $\Phi \in \Gamma(\mathfrak{S}^{\mathrm{reg}})$ of*

   $$\mathfrak{S}^{\mathrm{reg}} := (\mathfrak{s} \times Q) \times_{\mathrm{Sp}(1)\times H} S^{\mathrm{reg}}$$

   satisfying

   $$\mu(\Phi) = 0 \quad \text{and} \quad s = p \circ \Phi.$$

   Q and $Q \times_H K \cong R$ are unique up to isomorphism, and every two lifts Φ are related by a unique gauge transformation in $\mathscr{G}(P)$.

2. *Suppose $B \in \mathscr{A}(R)$. If $\Phi \in \Gamma(\mathfrak{S}^{\mathrm{reg}})$ satisfies $\mu(\Phi) = 0$, then there is a unique $A \in \mathscr{A}_B(Q)$ such that $\nabla_A \Phi \in \Omega^1(M, \mathfrak{H}_\Phi)$. In particular, for this connection*

   $$p_*(\not{D}_A \Phi) = \mathfrak{F}(s).$$

3. *The condition $p_*(\not{D}_A \Phi) = \mathfrak{F}(s)$ characterizes $A \in \mathscr{A}_B(Q)$ uniquely.*

Proof. Part (1) is proved by observing that the lifts exists locally and that the obstruction to the local lifts patching defines a cocycle which determines Q; see [Hay12] for details.

We prove (2). For an arbitrary connection $A_0 \in \mathscr{A}_B(Q)$ and for all $x \in M$, we have

$$(\nabla_{A_0} \Phi)(x) \in T_x^* M \otimes T_{\Phi(x)}(S^{\mathrm{reg}} \cap \mu^{-1}(0)).$$

By Proposition 2.1.5(2) there exists a unique $a \in \Omega^1(M, \mathfrak{g}_P)$ such that

$$\nabla_{A_0 + a} \Phi \in \Omega^1(M, \mathfrak{H}_\Phi).$$

The assertion in (2) now follows from the fact that for $s = p \circ \Phi$ we have $p_*(\nabla_{A_0} \Phi) = \nabla_B s$ and the definitions of $\not{D}_A$ and $\mathfrak{F}$.

We prove (3). If $a \in \Omega^1(M, \mathfrak{g}_P)$ and $A + a$ also satisfies this condition, then we must have

$$\bar{\gamma}(a)\Phi = 0.$$

This is impossible because $\Phi \in \Gamma(\mathfrak{S}^{\mathrm{reg}})$, that is, $(\mathrm{d}\mu)_\Phi$ is surjective; hence, its adjoint $\bar{\gamma}(\cdot)\Phi$ is injective. □

Proposition 2.5.9. *Given a set of geometric data as in Definition 2.2.1, let*

$$R := Q \times_H K, \quad \mathfrak{X} := (\mathfrak{s} \times R) \times_{\mathrm{Sp}(1)\times K} X, \quad \text{and} \quad \mathfrak{S}^{\mathrm{reg}} := (\mathfrak{s} \times Q) \times_{\mathrm{Sp}(1)\times H} S^{\mathrm{reg}}.$$

The map

$$\Gamma(\mu^{-1}(0) \cap \mathfrak{S}^{\mathrm{reg}})/\mathscr{G}(P) \to \Gamma(\mathfrak{X})$$

$$[\Phi] \mapsto p \circ \Phi$$

is a homeomorphism onto its image.

Proof. Fix $\Phi_0 \in \Gamma(\mu^{-1}(0) \cap \mathfrak{S}^{\mathrm{reg}})$ and set $s_0 := p \circ \Phi_0 \in \Gamma(\mathfrak{X})$. Given $0 < \sigma \ll 1$, for every $\Phi \in \Gamma(\mu^{-1}(0) \cap \mathfrak{S}^{\mathrm{reg}})$ with $\|\Phi - \Phi_0\|_{L^\infty} < \sigma$, there is a unique $u \in \mathscr{G}(P)$ such that

$$u\Phi \perp \operatorname{im}\big(\rho(\cdot)\Phi_0 \colon \Gamma(\mathfrak{g}_P) \to \Gamma(\mathfrak{S})\big);$$

moreover, for every $k \in \mathbf{N}$,

$$\|u\Phi - \Phi_0\|_{C^k} \lesssim_k \|\Phi - \Phi_0\|_{C^k}.$$

Thus, it suffices to show that the map

$$\left\{ \Phi \in \Gamma(\mu^{-1}(0) \cap \mathfrak{S}^{\mathrm{reg}}) : \begin{array}{l} \|\Phi - \Phi_0\|_{L^\infty} < \sigma \text{ and} \\ \Phi \perp \operatorname{im}\big(\rho(\cdot)\Phi_0 \colon \Gamma(\mathfrak{g}_P) \to \Gamma(\mathfrak{S})\big) \end{array} \right\} \to \Gamma(\mathfrak{X})$$

is a homeomorphism onto its image. This, however, is immediate from the Implicit Function Theorem and the fact that the tangent space at Φ_0 to the former space is $\Gamma(\mathfrak{H}_{\Phi_0})$ and the derivative of this map is the canonical isomorphism $\Gamma(\mathfrak{H}_{\Phi_0}) \cong \Gamma(s_0^* V\mathfrak{X})$ from Proposition 2.1.5(2). □

In the situation of Proposition 2.5.8, we have $|\Phi| = |\hat{v} \circ s|$. The preceding results thus imply the following.

Corollary 2.5.10. *Let R be a principal K–bundle. Set $\mathfrak{X} := R \times_K X$ and*

$$\mathfrak{M}_F := \{(\mathbf{p}, s) \in \mathscr{P} \times \Gamma(\mathfrak{X}) : \mathfrak{F}(s) = 0 \text{ and } \|\hat{v} \circ s\|_{L^2} = 1\}.$$

The map

$$\coprod_Q \partial\mathfrak{M}_{\mathrm{SW},Q} \to \mathfrak{M}_F$$

defined by

$$(\mathbf{p}, [(\Phi, A)]) \mapsto (\mathbf{p}, p \circ \Phi)$$

is a homeomorphism. Here, the disjoint union is taken over all isomorphism classes of principal H–bundles Q with isomorphisms $Q \times_H K \cong R$.

For future reference, we include here also three technical results about lifting infinitesimal deformations of Fueter sections to sections of $\mathfrak{S}$. These results, which can be seen as an infinitesimal version of the Haydys correspondence, will be important for studying deformation theory of Fueter sections.

Proposition 2.5.11. *For $\Phi \in \Gamma(\mu^{-1}(0) \cap \mathfrak{S}^{\mathrm{reg}})$, set $s := p \circ \Phi$ and let $A \in \mathscr{A}_B(Q)$ be as in Proposition 2.5.8. The isomorphism $p_* \colon \Gamma(\mathfrak{H}_\Phi) \to \Gamma(s^* V\mathfrak{X})$ identifies $\pi_{\mathfrak{H}} \nabla_A \colon \Omega^0(M, \mathfrak{H}_\Phi) \to \Omega^1(M, \mathfrak{H}_\Phi)$ with $\nabla_B \colon \Omega^0(M, s^* V\mathfrak{X}) \to \Omega^1(M, s^* V\mathfrak{X})$.*

Proof. If (Φ_t) is a one-parameter family of local sections of $\mu^{-1}(0) \cap \mathfrak{S}^{\mathrm{reg}}$ with

$$(\partial_t \Phi_t)|_{t=0} = \phi,$$

A_t are as in Proposition 2.5.8, and $a = (\partial_t A_t)|_{t=0}$, then we have

$$\partial_t \left(\pi_{\mathfrak{H}_{\Phi_t}} \nabla_{A_t} \Phi_t \right)\Big|_{t=0} = \left(\partial_t \pi_{\mathfrak{H}_{\Phi_t}}\right)\Big|_{t=0} \nabla_{A_0} \Phi_0 + \pi_{\mathfrak{H}_{\Phi_0}}(\rho(a)\Phi_0) + \pi_{\mathfrak{H}_{\Phi_0}}(\nabla_{A_0}\phi).$$

The first term vanishes because $\nabla_{A_0}\Phi_0 \in \Gamma(\mathfrak{H}_{\Phi_0})$, and the second term vanishes because of Proposition 2.1.5(2). □

If $\Phi \in \Gamma(\mu^{-1}(0) \cap \mathfrak{S}^{\mathrm{reg}})$, then the induced splitting $\mathfrak{S} = \mathfrak{H}_\Phi \oplus \mathfrak{N}_\Phi$ given by Proposition 2.1.5(2) need not be parallel for A as in Proposition 2.5.8.

Definition 2.5.12. The *second fundamental forms* of the splitting $\mathfrak{H}_\Phi \oplus \mathfrak{N}_\Phi$ are defined by

$$\begin{aligned} II &:= \pi_{\mathfrak{N}} \nabla_A \in \Omega^1(M, \mathrm{Hom}(\mathfrak{H}_\Phi, \mathfrak{N}_\Phi)) \quad \text{and} \\ II^* &:= -\pi_{\mathfrak{H}} \nabla_A \in \Omega^1(M, \mathrm{Hom}(\mathfrak{N}_\Phi, \mathfrak{H}_\Phi)). \end{aligned}$$

We decompose the Dirac operator $\not{D}_A$ according to $\mathfrak{S} = \mathfrak{H}_\Phi \oplus \mathfrak{N}_\Phi$ as

$$\not{D}_A = \begin{pmatrix} \not{D}_{\mathfrak{H}} & -\gamma II^* \\ \gamma II & \not{D}_{\mathfrak{N}} \end{pmatrix} \tag{2.5.4}$$

with

$$\begin{aligned} \not{D}_{\mathfrak{H}} &:= \gamma(\pi_{\mathfrak{H}} \nabla_A)\colon \Gamma(\mathfrak{H}_\Phi) \to \Gamma(\mathfrak{H}_\Phi) \quad \text{and} \\ \not{D}_{\mathfrak{N}} &:= \gamma(\pi_{\mathfrak{N}} \nabla_A)\colon \Gamma(\mathfrak{N}_\Phi) \to \Gamma(\mathfrak{N}_\Phi). \end{aligned}$$

The following result helps to better understand the off-diagonal terms in (2.5.4).

Proposition 2.5.13. *Suppose* $\Phi \in \Gamma(\mu^{-1}(0) \cap \mathfrak{S}^{\mathrm{reg}})$ *and* $\not{D}_A \Phi = 0$. *Writing* $\phi \in \Gamma(\mathfrak{N}_\Phi)$ *as*

$$\phi = \rho(\xi)\Phi + \bar{\gamma}(a)\Phi$$

for $\xi \in \Gamma(\mathfrak{g}_P)$ *and* $a \in \Omega^1(M, \mathfrak{g}_P)$, *we have*

$$-\gamma II^* \phi = 2 \sum_{i=1}^{3} \pi_{\mathfrak{H}} \left(\rho(a(e_i)) \nabla^A_{e_i} \Phi \right).$$

Here (e_1, e_2, e_3) *is a local orthonormal frame.*

Proof. Since $\nabla \Phi \in \Omega^1(M, \mathfrak{H}_\Phi)$ and $\not{D}_A \Phi = 0$, we have

$$\begin{aligned} -\gamma II^*(\rho(\xi)\Phi + \bar{\gamma}(a)\Phi_0) &= \sum_{i=1}^{3} \gamma(e^i) \pi_{\mathfrak{H}} \left(\rho(\xi) \nabla^A_{e_i} \Phi + \bar{\gamma}(a) \nabla^A_{e_i} \Phi \right) \\ &= \sum_{i=1}^{3} \pi_{\mathfrak{H}} \left((\gamma(e^i)\bar{\gamma}(a) + \bar{\gamma}(a)\gamma(e^i)) \nabla^A_{e_i} \Phi \right) \\ &= 2 \sum_{i=1}^{3} \pi_{\mathfrak{H}} (\rho(a(e_i)) \nabla^A_{e_i} \Phi). \end{aligned}$$

□

Proposition 2.5.14. *The isomorphism* $p_*\colon \Gamma(\mathfrak{H}_\Phi) \to \Gamma(s^* V\mathfrak{X})$ *identifies the linearized Fueter operator* $(\mathrm{d}\mathfrak{F})_s\colon \Gamma(s^* V\mathfrak{X}) \to \Gamma(s^* V\mathfrak{X})$ *with* $\not{D}_{\mathfrak{H}}\colon \Gamma(\mathfrak{H}_\Phi) \to \Gamma(\mathfrak{H}_\Phi)$.

Proof. The linearized Fueter operator is given by

$$(\mathrm{d}\mathfrak{F})_s \hat{s} = \gamma(\nabla_B \hat{s})$$

The assertion thus follows from Proposition 2.1.5(4) and Proposition 2.5.11. □

2.6 DEFORMATION THEORY OF FUETER SECTIONS

The main result of this section is a Kuranishi description of the space of Fueter sections.

Proposition 2.6.1. *Let $s_0 \in \Gamma(\mathfrak{X})$ be a Fueter section with respect to $\mathbf{p}_0 = (g_0, B_0) \in \mathscr{P}$. Denote by $\mathfrak{c}_0 \in \Gamma(\mathfrak{S}^{\mathrm{reg}}) \times \mathscr{A}(P)$ a lift of s_0. There exist an open neighbourhood U of $\mathbf{p}_0 \in \mathscr{P}$, an open neighborhood*

$$\mathscr{I}_\partial \subset I_\partial := \ker(\mathrm{d}\mathfrak{F})_{s_0} \cap (\hat{v} \circ s)^\perp$$

of 0, a smooth map

$$\mathrm{ob}_\partial \colon U \times \mathscr{I}_\partial \to \mathrm{coker}(\mathrm{d}\mathfrak{F})_{s_0},$$

an open neighborhood V of $([\mathbf{p}_0, \mathfrak{c}_0]) \in \partial\mathfrak{M}_{\mathrm{SW}}$, and a homeomorphism

$$\mathfrak{x}_\partial \colon \mathrm{ob}_\partial^{-1}(0) \to V \subset \partial\mathfrak{M}_{\mathrm{SW}}$$

which maps $(\mathbf{p}_0, 0)$ to $(\mathbf{p}_0, 0, [\mathfrak{c}_0])$ and commutes with the projections to $\mathscr{P}$.

Since $\partial\mathfrak{M}_{\mathrm{SW}} \cong \mathfrak{M}_F$ through the Haydys correspondence, this result has a straightforward proof using Lemma 2.3.13, which makes no reference to the Seiberg–Witten equation. However, this is not the approach we take because our principal goal, which will be achieved in the next chapter, is to compare the deformation theory of Fueter sections with that of solutions of the Seiberg–Witten equation. Thus, in the remaining part of this section we construct the Kuranishi model by analyzing equation (2.5.1) with $\varepsilon = 0$ rather than the Fueter equation. The main difficulty in this construction is that (2.5.1) is not an elliptic equation for $\varepsilon = 0$, even modulo gauge; indeed, it does not include a differential equation for the connection A.

Fix $k \in \mathbf{N}$ and $p \in (1, \infty)$ with $(k+1)p > 3$. Let

$$\partial\mathfrak{M}^{k,p}_{\mathrm{SW}} = \left\{ (\mathbf{p}, [(\Phi, A)]) \in \mathscr{P} \times \frac{W^{k+1,p}\Gamma(\mathfrak{S}) \times W^{k,p}\mathscr{A}(Q)}{W^{k+1,p}\mathscr{G}(P)} : \begin{array}{l} A \text{ induces } B, \\ (\Phi, A) \text{ satisfies } (2.5.1) \\ \text{with } \varepsilon = 0, \\ \text{and } \|\Phi\|_{L^2} = 1 \end{array} \right\}.$$

By the Haydys correspondence $\partial\mathfrak{M}^{k,p}_{\mathrm{SW}}$ is homeomorphic to $\mathfrak{M}^{k,p}_F$, the universal moduli space of normalized $W^{k+1,p}$ Fueter sections of $\mathfrak{X}$. Consequently, for $\ell \in \mathbf{N}$ and $q \in (1, \infty)$ with $\ell \geqslant k$ and $q \geqslant p$, the inclusions $\partial\mathfrak{M}^{\ell,q}_{\mathrm{SW}} \subset \partial\mathfrak{M}^{k,p}_{\mathrm{SW}} \subset \partial\mathfrak{M}_{\mathrm{SW}}$ are homeomorphisms; see also Proposition 2.6.5.

Proposition 2.6.2. *Assume the situation of Proposition 2.6.1. For $\mathbf{p} \in \mathscr{P}$, set*

$$\begin{aligned} X_0 &:= W^{k+1,p}\Gamma(\mathfrak{S}) \oplus W^{k,p}\Omega^1(M, \mathfrak{g}_P) \oplus W^{k,p}\Omega^0(M, \mathfrak{g}_P) \\ \textit{and} \quad Y &:= W^{k,p}\Gamma(\mathfrak{S}) \oplus W^{k+1,p}\Omega^1(M, \mathfrak{g}_P) \oplus W^{k+1,p}\Omega^0(M, \mathfrak{g}_P) \oplus \mathbf{R}, \end{aligned}$$

and define a linear map $L_{\mathbf{p},0}\colon X_0 \to Y$*, a quadratic map* $Q_{\mathbf{p},0}\colon X_0 \to Y$*, and* $\mathfrak{e}_{\mathbf{p},0} \in Y$ *by*

$$L_{\mathbf{p},0} := \begin{pmatrix} -\not{D}_{A_0} & -\bar{\gamma}(\cdot)\Phi_0 & -\rho(\cdot)\Phi_0 \\ -2 * \mu(\Phi_0, \cdot) & 0 & 0 \\ -\rho^*(\cdot\, \Phi_0^*) & 0 & 0 \\ 2\langle \cdot, \Phi_0 \rangle_{L^2} & 0 & 0 \end{pmatrix},$$

$$Q_{\mathbf{p},0}(\phi, a, \xi) := \begin{pmatrix} -\bar{\gamma}(a)\phi \\ - * \mu(\phi) \\ 0 \\ \|\phi\|_{L^2}^2 \end{pmatrix}, \quad \text{and} \quad \mathfrak{e}_{\mathbf{p},0} := \begin{pmatrix} -\not{D}_{A_0}\Phi_0 \\ -\mu(\Phi_0) \\ 0 \\ \|\Phi_0\|_{L^2}^2 - 1 \end{pmatrix},$$

respectively.[4]

There exist a neighborhood U *of* $\mathbf{p}_0 \in \mathscr{P}$ *and* $\sigma > 0$*, such that, for every* $\mathbf{p} \in U$ *and* $\hat{\mathfrak{c}} = (\phi, a, \xi) \in B_\sigma(0) \subset X_0$*, we have*

$$L_{\mathbf{p},0}\hat{\mathfrak{c}} + Q_{\mathbf{p},0}(\hat{\mathfrak{c}}) + \mathfrak{e}_{\mathbf{p},0} = 0 \tag{2.6.1}$$

if and only if $\xi = 0$ *and* $(\Phi, A) = (\Phi_0 + \phi, A_0 + a)$ *satisfies*

$$\not{D}_A\Phi = 0 \quad \text{and} \quad \mu(\Phi) = 0 \tag{2.6.2}$$

as well as

$$\|\Phi\|_{L^2} = 1 \quad \text{and} \quad \rho^*(\Phi\Phi_0^*) = 0.$$

Remark 2.6.3. The above proposition engages in the following abuse of notation. If $A_0 \in \mathscr{A}_B(Q)$ and $B' \in \mathscr{A}(R)$, then $b = B' - B \in \Omega^1(M, \mathfrak{g}_R)$. Since $\mathrm{Lie}(K) = \mathfrak{g}^\perp \subset \mathfrak{h}$ we have a map $\Omega^1(M, \mathfrak{g}_R) \to \Omega^1(M, \mathfrak{g}_Q)$ and can identify $A_0 \in \mathscr{A}_B(Q)$ with "A_0" $= A_0 + b \in \mathscr{A}_{B'}(Q)$.

Together with (the argument from the proof of) Proposition 2.5.9 we obtain the following.

Corollary 2.6.4. *Assume the situation of Proposition 2.6.1. With* $U \subset \mathscr{P}$ *and* $\sigma > 0$ *as in Proposition 2.6.2, the map*

$$\{(\mathbf{p}, \hat{\mathfrak{c}}) \in U \times B_\sigma(0) \text{ satisfying } (2.6.1)\} \to \partial\mathfrak{M}_{\mathrm{SW}}$$

defined by

$$(\mathbf{p}, \phi, a, \xi) \mapsto (\mathbf{p}, [(\Phi_0 + \phi, A_0 + a)])$$

is a homeomorphism onto a neighborhood of $[\mathfrak{c}_0]$*.*

Proof of Proposition 2.6.2. If $\hat{\mathfrak{c}} = (\phi, a, \xi)$ satisfies (2.6.1), then $\Phi = \Phi_0 + \phi$ and $A = A_0 + a$ satisfy

$$\not{D}_A\Phi + \rho(\xi)\Phi_0 = 0, \quad \mu(\Phi) = 0, \quad \text{and} \quad \rho^*(\phi\Phi_0^*) = 0.$$

4 The term $\mathfrak{e}_{\mathbf{p},0}$ vanishes for $\mathbf{p} = \mathbf{p}_0$.

Hence, by Proposition 2.3.9,

$$0 = \mathrm{d}_A \mu(\Phi) = -\rho(\not{D}_A \Phi\Phi^*) = \rho^*(\rho(\xi)\Phi_0(\Phi_0 + \phi)) = R^*_{\Phi_0} R_{\Phi_0}\xi + O(|\xi||\phi|)$$

with

$$R_{\Phi_0} := \rho(\cdot)\Phi_0.$$

Since Φ_0 is regular, R_{Φ_0} is injective, and it follows that $\xi = 0$ if $|\phi| \lesssim \sigma \ll 1$ and $\mathbf{p}$ is sufficiently close to $\mathbf{p}_0$. □

Proof of Proposition 2.6.1. Denote by $\iota\colon \operatorname{coker}(\mathrm{d}\mathfrak{F})_{s_0} \cong \operatorname{coker}\not{D}_{\mathfrak{H}} \to \Gamma(\mathfrak{H})$ the inclusion of the L^2 orthogonal complement of $\operatorname{im}\not{D}_{\mathfrak{H}}$. Denote by $\pi_0\colon \Gamma(\mathfrak{H}) \to I_\partial$ the L^2 orthogonal projection onto $I_\partial = \ker(\mathrm{d}\mathfrak{F})_{s_0} \cap (\hat{v} \circ s)^\perp \subset \ker \not{D}_{\mathfrak{H}} \cong \ker(\mathrm{d}\mathfrak{F})_{s_0}$. Define

$$\bar{\not{D}}_{\mathfrak{H}}\colon\ \operatorname{coker}(\mathrm{d}\mathfrak{F})_{s_0} \oplus \Gamma(\mathfrak{H}) \to I_\partial \oplus \mathbf{R} \oplus \Gamma(\mathfrak{H})$$

by

$$\bar{\not{D}}_{\mathfrak{H}} := \begin{pmatrix} 0 & \pi_0 \\ 0 & -2\langle\cdot, \Phi_0\rangle_{L^2} \\ \iota & \not{D}_{\mathfrak{H}} \end{pmatrix}.$$

Set

$$\begin{aligned} \bar{X}_0 &:= \operatorname{coker}(\mathrm{d}\mathfrak{F})_{s_0} \oplus W^{k+1,p}\Gamma(\mathfrak{H}) \\ &\quad \oplus W^{k+1,p}\Gamma(\mathfrak{N}) \\ &\quad \oplus W^{k,p}\Omega^1(M, \mathfrak{g}_P) \oplus W^{k,p}\Omega^0(M, \mathfrak{g}_P) \quad \text{and} \\ \bar{Y} &:= I_\partial \oplus \mathbf{R} \oplus W^{k,p}\Gamma(\mathfrak{H}) \\ &\quad \oplus W^{k,p}\Gamma(\mathfrak{N}) \\ &\quad \oplus W^{k+1,p}\Omega^1(M, \mathfrak{g}_P) \oplus W^{k+1,p}\Omega^0(M, \mathfrak{g}_P). \end{aligned} \tag{2.6.3}$$

Define the operator $\bar{L}_{\mathbf{p},0}\colon\ \bar{X}_0 \to \bar{Y}$ by

$$\bar{L}_{\mathbf{p},0} := \begin{pmatrix} -\bar{\not{D}}_{\mathfrak{H}} & \gamma II^* & 0 & \\ -\gamma II & -\not{D}_{\mathfrak{N}} & -\mathfrak{a} & 0 \\ 0 & -\mathfrak{a}^* & 0 & \end{pmatrix} \tag{2.6.4}$$

with $\mathfrak{a}\colon\ \Omega^1(M, \mathfrak{g}_P) \oplus \Omega^0(M, \mathfrak{g}_P) \to \Gamma(\mathfrak{N})$ defined by

$$\mathfrak{a}(a, \xi) := \bar{\gamma}(a)\Phi_0 + \rho(\xi)\Phi.$$

The operator $\bar{\not{D}}_{\mathfrak{H}}$ is invertible because

$$\begin{pmatrix} \pi_0 \\ -2\langle\cdot, \Phi_0\rangle_{L^2} \end{pmatrix}$$

is essentially the L^2 orthogonal projection onto $\ker \not{D}_{\mathfrak{H}}$. It can be verified by a direct computation that $\bar{L}_{\mathbf{p}_0,0}$ is invertible and its inverse is given by

$$\begin{pmatrix} -\bar{\not{D}}_{\mathfrak{H}}^{-1} & 0 & -\bar{\not{D}}_{\mathfrak{H}}^{-1}\gamma II^*(\mathfrak{a}^*)^{-1} \\ 0 & 0 & -(\mathfrak{a}^*)^{-1} \\ \mathfrak{a}^{-1}\gamma II\bar{\not{D}}_{\mathfrak{H}}^{-1} & -\mathfrak{a}^{-1} & \mathfrak{a}^{-1}\not{D}_{\mathfrak{N}}(\mathfrak{a}^*)^{-1} + \mathfrak{a}^{-1}\gamma II\bar{\not{D}}_{\mathfrak{H}}^{-1}\gamma II^*(\mathfrak{a}^*)^{-1} \end{pmatrix}. \tag{2.6.5}$$

After possibly shrinking U, we can assume that $\bar{L}_{\mathbf{p},0}$ is invertible for every $\mathbf{p} \in U$.
Since $Q_{\mathbf{p},0}$ is a quadratic map and

$$\begin{aligned}\|Q_{\mathbf{p},0}(\phi, a, \xi)\|_Y &= \|\bar{\gamma}(a)\phi\|_{W^{k,p}} + \|\mu(\phi)\|_{W^{k+1,p}} + \|\phi\|_{L^2}^2 \\ &\lesssim \|a\|_{W^{k,p}}\|\phi\|_{W^{k+1,p}} + \|\phi\|_{W^{k+1,p}}^2,\end{aligned} \tag{2.6.6}$$

$Q_{\mathbf{p},0}$ satisfies (2.3.4); hence, we can apply Lemma 2.3.13 to complete the proof. □

In the following regularity result, we decorate X_0 and Y with superscripts indicating the choice of the differentiability and integrability parameters k and p.

Proposition 2.6.5. *Assume the situation of Proposition 2.6.1. For each $k, \ell \in \mathbf{N}$ and $p, q \in (1, \infty)$ with $(k+1)p > 3$, $\ell \geqslant k$, and $q \geqslant p$, there are constants $c, \sigma > 0$ and an open neighborhood U of $\mathbf{p}_0$ in $\mathscr{P}$ such that if $\mathbf{p} \in U$ and $\hat{\mathfrak{c}} \in B_\sigma(0) \subset X_0^{k,p}$ is solution of*

$$L_{\mathbf{p},0}\hat{\mathfrak{c}} + Q_{\mathbf{p},0}(\hat{\mathfrak{c}}) + \mathfrak{e}_{\mathbf{p},0} = 0,$$

then $\hat{\mathfrak{c}} \in X_0^{\ell,q}$ and $\|\hat{\mathfrak{c}}\|_{X_0^{\ell,q}} \leqslant c\|\hat{\mathfrak{c}}\|_{X_0^{k,p}}$.

Proof. Provided U is a sufficiently small neighborhood of $\mathbf{p}_0$ and $0 < \sigma \ll 1$, it follows from Banach's Fixed Point Theorem that $(0, \hat{\mathfrak{c}})$ is the unique solution in $B_\sigma(0) \subset \bar{X}^{k,p}$ of

$$\bar{L}_{\mathbf{p},0}(0, \hat{\mathfrak{c}}) + Q_{\mathbf{p},0}(\hat{\mathfrak{c}}) + \mathfrak{e}_{\mathbf{p},0} = \begin{pmatrix}\pi\hat{\mathfrak{c}} \\ 0\end{pmatrix},$$

and that there exists a $(o, \hat{\mathfrak{d}}) \in B_\sigma(0) \subset \bar{X}^{\ell,q}$ such that

$$\bar{L}_{\mathbf{p},0}(o, \hat{\mathfrak{d}}) + Q_{\mathbf{p},0}(\hat{\mathfrak{d}}) + \mathfrak{e}_{\mathbf{p},0} = \begin{pmatrix}\pi\hat{\mathfrak{c}} \\ 0\end{pmatrix}.$$

Since $\bar{X}^{\ell,q} \subset \bar{X}^{k,p}$ and $\|(o, \hat{\mathfrak{d}})\|_{\bar{X}^{k,p}} \leqslant \|(o, \hat{\mathfrak{d}})\|_{\bar{X}^{\ell,q}} \leqslant \sigma$, it follows that $(o, \hat{\mathfrak{d}}) = (0, \hat{\mathfrak{c}})$ and thus $\hat{\mathfrak{c}} \in \bar{X}^{\ell,q}$ and $\|\hat{\mathfrak{c}}\|_{X^{\ell,q}} \leqslant \sigma$. From this it follows easily that $\|\hat{\mathfrak{c}}\|_{X^{\ell,q}} \leqslant c\|\hat{\mathfrak{c}}\|_{X^{k,p}}$. □

2.7 GLUING KURANISHI MODELS

In Section 2.3 we constructed Kuranishi models for the universal Seiberg–Witten moduli space $\mathfrak{M}_{\mathrm{SW}}$. Similarly, in Section 2.6 we constructed Kuranishi models for the universal moduli space of Fueter sections $\partial\mathfrak{M}_{\mathrm{SW}}$. In this section, we build Kuranishi models for the universal partially compactified moduli space $\overline{\mathfrak{M}}_{\mathrm{SW}}$ centered at points of $\partial\mathfrak{M}_{\mathrm{SW}}$ (see Definition 2.5.3 for the relevant definitions).

Unlike the previous results, this construction is not a standard application of the Inverse Function Theorem in Banach spaces. The difficulty arises from the fact that the (gauge fixed and co-gauged fixed) linearization of (2.5.1) appears to become degenerate as ε approaches zero. The Haydys correspondence, however, indicates that one can reinterpret (2.5.1) at $\varepsilon = 0$ as the Fueter equation; in particular, as a non-degenerate elliptic PDE. One can think of the main result of this section, Theorem 2.7.1 below, as a gluing theorem for the Kuranishi model described in Proposition 2.3.1 with a Kuranishi model for the moduli space of Fueter sections divided by the $\mathbf{R}^+$–action.

Theorem 2.7.1. *Let* $\mathbf{p}_0 = (g_0, B_0) \in \mathscr{P}$ *and* $\mathfrak{c}_0 = (\Phi_0, A_0) \in \Gamma(\mathfrak{S}^{\mathrm{reg}}) \times \mathscr{A}_B(Q)$ *be such that* $(\mathbf{p}_0, 0, [\mathfrak{c}_0]) \in \partial\mathfrak{M}_{\mathrm{SW}}$. *Denote by* $s_0 = p \circ \Phi_0 \in \Gamma(\mathfrak{X})$ *the corresponding Fueter section of* $\mathfrak{X}$. *Set*

$$I_\partial := \ker(\mathrm{d}\mathfrak{F})_{s_0} \cap (\hat{\vartheta} \circ s)^\perp \quad \textit{and} \quad O := \operatorname{coker}(\mathrm{d}\mathfrak{F})_{s_0}.$$

Let $r \in \mathbf{N}$.

There exist an open neighborhood $\mathscr{I}_\partial$ *of* $0 \in I_\partial$, *a constant* $\varepsilon_0 > 0$, *an open neighborhood* $U \subset \mathscr{P}$ *of* $\mathbf{p}_0$, *a* C^{2r-1} *map*

$$\mathrm{ob}\colon U \times [0, \varepsilon_0) \times \mathscr{I}_\partial \to O,$$

an open neighborhood V *of* $(\mathbf{p}_0, 0, [\mathfrak{c}_0]) \in \overline{\mathfrak{M}}_{\mathrm{SW}}$, *and a homeomorphism*

$$\mathfrak{x}\colon \mathrm{ob}^{-1}(0) \to V \subset \overline{\mathfrak{M}}_{\mathrm{SW}}$$

such that the following hold:

1. *There are smooth functions*

$$\mathrm{ob}_\partial, \widehat{\mathrm{ob}}_1, \dots, \widehat{\mathrm{ob}}_r\colon U \times \mathscr{I}_\partial \to O$$

such that for all $m, n \in \mathbf{N}_0$ *with* $m + n \leqslant 2r$ *we have*

$$\left\| \nabla^m_{U \times \mathscr{I}_\partial} \partial^n_\varepsilon \left(\mathrm{ob} - \mathrm{ob}_\partial - \textstyle\sum_{i=1}^r \varepsilon^{2i} \widehat{\mathrm{ob}}_i \right) \right\|_{C^0} = O(\varepsilon^{2r-n+2}).$$

2. *The map* $\mathfrak{x}$ *commutes with the projection to* $\mathscr{P} \times [0, \infty)$ *and satisfies*

$$\mathfrak{x}(\mathbf{p}_0, 0, 0) = (\mathbf{p}_0, 0, [\mathfrak{c}_0]).$$

3. *For each* $(\mathbf{p}, \mathfrak{c}) \in \operatorname{im} \mathfrak{x} \cap \mathfrak{M}_{\mathrm{SW}}$, *the solution* $\mathfrak{c}$ *is irreducible; moreover, it is unobstructed if* $\mathrm{d}_I \mathrm{ob}$ *is surjective.*

Remark 2.7.2. The neighborhoods $\mathscr{I}_\partial$ and U may depend on the choice of r.

In the remaining part of this section we prove Theorem 2.7.1, whose hypotheses we will assume throughout.

Fix $k \in \mathbf{N}$ and $p \in (1, \infty)$ with $(k+1)p > 3$. Let

$$\mathfrak{M}^{k,p}_{\mathrm{SW}} = \left\{ (\mathbf{p}, \varepsilon, [(\Phi, A)]) \in \mathscr{P} \times \mathbf{R}^+ \times \frac{W^{k+1,p}\Gamma(\mathfrak{S}) \times W^{k+2,p}\mathscr{A}(P)}{W^{k+3,p}\mathscr{G}(P)} : (\varepsilon, \Phi, A) \text{ satisfies } (2.5.1) \right\}.$$

For $\ell \in \mathbf{N}$ and $q \in (1, \infty)$ with $\ell \geqslant k$ and $q \geqslant p$, the inclusions $\mathfrak{M}^{\ell,q}_{\mathrm{SW}} \subset \mathfrak{M}^{k,p}_{\mathrm{SW}} \subset \mathfrak{M}_{\mathrm{SW}}$ are homeomorphisms; see also Proposition 2.7.9.

2.7.1 *Reduction to a slice*

Proposition 2.7.3. *Let* $\mathfrak{c}_0 = (\Phi_0, A_0) \in \Gamma(\mathfrak{S}^{\mathrm{reg}}) \times \mathscr{A}(P)$ *and* $\mathbf{p}_0 \in \mathscr{P}$. *For* $\mathbf{p} \in \mathscr{P}$, *set*

$$X_\varepsilon := W^{k+1,p}\Gamma(\mathfrak{S}) \oplus W^{k+2,p}\Omega^1(M, \mathfrak{g}_P) \oplus W^{k+2,p}\Omega^0(M, \mathfrak{g}_P)$$

and

$$\|(\phi, a, \xi)\|_{X_\varepsilon} := \|\phi\|_{W^{k+1,p}} + \|(a,\xi)\|_{W^{k,p}} + \varepsilon\|\nabla^{k+1}(a,\xi)\|_{L^p} + \varepsilon^2\|\nabla^{k+2}(a,\xi)\|_{L^p}.$$

There exist a neighborhood U of $\mathbf{p}_0 \in \mathscr{P}$ and constants $\sigma, \varepsilon_0, c > 0$ such that the following holds. If $\mathbf{p} \in U$, $\hat{\mathfrak{c}} = (\phi, a) \in X_\varepsilon$, and $\varepsilon \in (0, \varepsilon_0]$ are such that

$$\|\hat{\mathfrak{c}}\|_{X_\varepsilon} < \sigma,$$

then there exists a $W^{k+3,p}$ gauge transformation g such that $(\tilde{\phi}, \tilde{a}) = g(\mathfrak{c}_0 + \hat{\mathfrak{c}}) - \mathfrak{c}_0$ satisfies

$$\|(\tilde{\phi}, \tilde{a})\|_{X_\varepsilon} < c\sigma,$$

and

$$\varepsilon^2 \mathrm{d}^*_{A_0 B}\tilde{a} - \rho^*(\tilde{\phi}\Phi_0^*) = 0. \tag{2.7.1}$$

Proof. To construct g, note that for $g = e^\xi$ with $\xi \in W^{k+3,p}\Omega^0(M, \mathfrak{g}_P)$ we have

$$\tilde{\phi} = \rho(\xi)\Phi_0 + \rho(\xi)\phi + \mathfrak{m}(\xi) \quad \text{and} \quad \tilde{a} = a - \mathrm{d}_{A_0}\xi - [a,\xi] + \mathfrak{n}(\xi).$$

Here $\mathfrak{n}$ and $\mathfrak{m}$ denote expressions which are algebraic and at least quadratic in ξ. The gauge fixing condition (2.7.1) can thus be written as

$$\mathfrak{l}_\varepsilon\xi + \mathfrak{d}_\varepsilon\xi + \mathfrak{q}_\varepsilon(\xi) + \mathfrak{e}_\varepsilon = 0.$$

with

$$\begin{aligned} \mathfrak{l}_\varepsilon &:= \varepsilon^2\Delta_{A_0 B} + R^*_{\Phi_0}R_{\Phi_0}, & \mathfrak{d}_\varepsilon &:= \varepsilon^2\mathrm{d}^*_{A_0 B}[a,\cdot] + \rho^*(\rho(\cdot)\phi\Phi_0^*), \\ \mathfrak{q}_\varepsilon(\xi) &:= \varepsilon^2\mathrm{d}^*_{A_0 B}\mathfrak{n}(\xi) + \rho^*(\mathfrak{m}(\xi)\Phi_0^*), & \mathfrak{e}_\varepsilon &:= -\varepsilon^2\mathrm{d}^*_{A_0 B}a - \rho^*(\phi\Phi_0). \end{aligned}$$

Denote by G_ε the Banach space $W^{k+3,p}\Omega^0(M, \mathfrak{g}_P)$ equipped with the norm

$$\|\xi\|_{G_\varepsilon} := \|\xi\|_{W^{k+1}} + \varepsilon\|\nabla^{k+2}\xi\|_{L^p} + \varepsilon^2\|\nabla^{k+3}\xi\|_{L^p}. \tag{2.7.2}$$

Since Φ_0 is regular, the operator $R^*_{\Phi_0}R_{\Phi_0}$ is positive definite; hence, for $\varepsilon \ll 1$, the operator

$$\mathfrak{l}_\varepsilon\colon G_\varepsilon \to W^{k+1,p}\Omega^0(M, \mathfrak{g}_P)$$

is invertible and $\|\mathfrak{l}_\varepsilon^{-1}\|_{\mathcal{L}(G_\varepsilon, W^{k+1,p})}$ is bounded independent of ε. Since

$$\|\mathfrak{d}_\varepsilon\|_{\mathcal{L}(G_\varepsilon, W^{k+1,p})} \lesssim \sigma \ll 1,$$

$\mathfrak{l}_\varepsilon + \mathfrak{d}_\varepsilon\colon G_\varepsilon \to W^{k+1,p}\Omega^0(M, \mathfrak{g}_P)$ will also be invertible with inverse bounded independent of ε and σ. Since the non-linearity $\mathfrak{q}_\varepsilon\colon G_\varepsilon \to W^{k+1,p}\Omega^0(M, \mathfrak{g}_P)$ satisfies (2.3.4) and $\|\mathfrak{e}_\varepsilon\| \lesssim \sigma \ll 1$, it follows from Banach's Fixed Point Theorem that, for a suitable $c > 0$, there exists a unique solution $\xi \in B_{c\sigma}(0) \subset G_\varepsilon$ to (2.7.2). This proves the existence of the desired gauge transformation, and local uniqueness. Global uniqueness follows by an argument by contradiction, cf. [DK90, Proposition 4.2.9]. □

Proposition 2.7.4. *Let $\mathfrak{c}_0 = (\Phi_0, A_0)$ be a lift of a Fueter section $s_0 \in \Gamma(\mathfrak{X})$ for $\mathbf{p}_0 \in \mathscr{P}$. Fix $\varepsilon > 0$ and $\mathbf{p} \in \mathscr{P}$. Define a linear map $L_{\mathbf{p},\varepsilon}\colon X_\varepsilon \to Y$ and a quadratic map $Q_{\mathbf{p},\varepsilon}\colon X_0 \to Y$ by*

$$L_{\mathbf{p},\varepsilon} := \begin{pmatrix} -\not{D}_{A_0} & -\gamma(\cdot)\Phi_0 & -\rho(\cdot)\Phi_0 \\ -2*\mu(\Phi_0,\cdot) & *\varepsilon^2 \mathrm{d}_{A_0} & \varepsilon^2 \mathrm{d}_{A_0} \\ -\rho^*(\cdot\,\Phi_0^*) & \varepsilon^2 \mathrm{d}_{A_0}^* & 0 \\ 2\langle \Phi_0, \cdot\rangle_{L^2} & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$Q_{\mathbf{p},\varepsilon}(\phi, a, \xi) := \begin{pmatrix} -\bar{\gamma}(a)\phi \\ \frac{1}{2}\varepsilon^2 * [a\wedge a] - *\mu(\phi) \\ 0 \\ \|\phi\|_{L^2}^2 \end{pmatrix},$$

respectively. With $\mathfrak{e}_{\mathbf{p},0}$ as in Proposition 2.6.2 set

$$\mathfrak{e}_{\mathbf{p},\varepsilon} := \mathfrak{e}_{\mathbf{p},0} + \varepsilon^2(0, *\varpi F_{A_0}, 0).$$

There exist a neighborhood U of $\mathbf{p}_0 \in \mathscr{P}$ and $\sigma > 0$ such that $\hat{\mathfrak{c}} = (\phi, a, \xi) \in B_\sigma(0) \subset X_\varepsilon$ satisfies

$$L_{p,\varepsilon}\hat{\mathfrak{c}} + Q_{\mathbf{p},\varepsilon}(\hat{\mathfrak{c}}) + \mathfrak{e}_{\mathbf{p},\varepsilon} = 0 \tag{2.7.3}$$

if and only if $\xi = 0$, $(A, \Phi) = (A_0 + a, \Phi_0 + \phi)$ satisfies

$$\not{D}_A \Phi = 0, \quad \varepsilon^2 \varpi F_A = \mu(\Phi), \quad \text{and} \quad \|\Phi\|_{L^2} = 1,$$

and

$$\varepsilon^2 \mathrm{d}_{A_0}^* a - \rho^*(\phi\Phi_0^*) = 0. \tag{2.7.4}$$

Proof. We only need to show that ξ vanishes, but this follows from the same argument as in the proof of Proposition 2.6.2 because $\mathrm{d}_A F_A = 0$. □

Corollary 2.7.5. *There exist $\varepsilon, \sigma > 0$ such the map*

$$\{(\mathbf{p}, \varepsilon, \phi, a, \xi) \in \mathscr{P} \times U \times (0, \varepsilon_0) \times B_\sigma(0) \text{ satisfying (2.7.3)}\} \to \mathfrak{M}_{\mathrm{SW}}$$

defined by

$$(\mathbf{p}, \varepsilon, \phi, a, \xi) \mapsto (\mathbf{p}, \varepsilon, [(\Phi_0 + \phi, A_0 + a)])$$

is a homeomorphism onto the intersection of $\mathfrak{M}_{\mathrm{SW}}$ with a neighborhood of $([\mathfrak{c}_0], \mathbf{p}_0, 0)$ in $\overline{\mathfrak{M}}_{\mathrm{SW}}$.

2.7.2 *Inverting $\bar{L}_{\mathbf{p},\varepsilon}$ for small ε*

For every $\varepsilon > 0$, define the Banach space $(\bar{X}_\varepsilon, \|\cdot\|_{\bar{X}_\varepsilon})$ by

$$\bar{X}_\varepsilon := \operatorname{coker}(\mathrm{d}\mathfrak{F})_{s_0} \oplus W^{k+1,p}\Gamma(\mathfrak{S}) \oplus W^{k+2,p}\Omega^1(M, \mathfrak{g}_P) \oplus W^{k+2,p}\Omega^0(M, \mathfrak{g}_P)$$

with norm

$$\|(o, \hat{\mathfrak{c}})\|_{\bar{X}_\varepsilon} := |o| + \|\hat{\mathfrak{c}}\|_{X_\varepsilon},$$

and the Banach space $(\bar{Y}, \|\cdot\|_{\bar{Y}})$ by

$$\bar{Y} := I_\partial \oplus \mathbf{R} \oplus W^{k,p}\Gamma(\mathfrak{S}) \oplus W^{k+1,p}\Omega^1(M, \mathfrak{g}_P) \oplus W^{k+1,p}\Omega^0(M, \mathfrak{g}_P)$$

with the obvious norm. Let $\bar{\not{D}}_{\mathfrak{H}}\colon \operatorname{coker}(\mathrm{d}\mathfrak{F})_{s_0} \oplus W^{k+1,p}\Gamma(\mathfrak{S}) \to I_\partial \oplus \mathbf{R} \oplus W^{k,p}\Gamma(\mathfrak{S})$ be as in the Proof of Proposition 2.6.1. Define $\bar{L}_{\mathbf{p},\varepsilon}\colon \bar{X}_\varepsilon \to \bar{Y}$ by

$$\bar{L}_{\mathbf{p},\varepsilon} := \begin{pmatrix} -\bar{\not{D}}_{\mathfrak{H}} & \gamma \mathrm{II}^* & 0 \\ -\gamma\mathrm{II} & -\not{D}_{\mathfrak{N}} & -\mathfrak{a} \\ 0 & -\mathfrak{a}^* & \varepsilon^2\delta_{A_0} \end{pmatrix} \tag{2.7.5}$$

with

$$\delta_{A_0} := \begin{pmatrix} *\mathrm{d}_{A_0} & \mathrm{d}_{A_0} \\ \mathrm{d}_{A_0}^* & 0 \end{pmatrix}.$$

Proposition 2.7.6. *There exist $\varepsilon_0, c > 0$, and a neighborhood U of $\mathbf{p}_0 \in \mathscr{P}$ such that, for all $\mathbf{p} \in U$ and $\varepsilon \in (0, \varepsilon_0]$, $\bar{L}_{\mathbf{p},\varepsilon}\colon \bar{X}_\varepsilon \to \bar{Y}$ is invertible, and $\left\|\bar{L}_{\mathbf{p},\varepsilon}^{-1}\right\| \leqslant c$.*

The proof of this result relies on the following two observations.

Proposition 2.7.7. *For $i = 1, 2, 3$, let V_i and W_i be Banach spaces, and set*

$$V := \bigoplus_{i=1}^{3} V_i \quad \text{and} \quad W := \bigoplus_{i=1}^{3} W_i.$$

Let $L\colon V \to W$ be a bounded linear operator of the form

$$L = \begin{pmatrix} D_1 & B_+ & 0 \\ B_- & D_2 & A_+ \\ 0 & A_- & D_3 \end{pmatrix}.$$

If the operators

$$D_1\colon V_1 \to W_1,$$
$$A_-\colon V_2 \to W_3, \quad \text{and}$$
$$Z := A_+ - (D_2 - B_- D_1^{-1} B_+) A_-^{-1} D_3\colon V_3 \to W_2$$

are invertible, then there exists a bounded linear operator $R\colon W \to V$ such that

$$RL = \mathrm{id}_W.$$

Moreover, the operator norm $\|R\|$ is bounded by a constant depending only on $\|L\|$, $\|D_1^{-1}\|$, $\|A_-^{-1}\|$, and $\|Z^{-1}\|$.

Proposition 2.7.8. *There exist $\varepsilon_0, c > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, the linear map*

$$\mathfrak{z}_\varepsilon := \mathfrak{a} + \varepsilon^2 \left(\not{D}_{\mathfrak{N}} + \gamma \mathrm{II} \not{D}_{\mathfrak{H}}^{-1} \gamma \mathrm{II}^* \right) (\mathfrak{a}^*)^{-1} \delta_{A_0}$$

acting between the spaces

$$W^{k+2,p}\Omega^1(M, \mathfrak{g}_P) \oplus W^{k+2,p}\Omega^0(M, \mathfrak{g}_P) \to W^{k,p}\Gamma(\mathfrak{N})$$

is invertible, and

$$\|\mathfrak{z}_\varepsilon^{-1}(a,\xi)\|_{W^{k,p}} + \varepsilon\|\nabla^{k+1}\mathfrak{z}_\varepsilon^{-1}(a,\xi)\|_{L^p} + \varepsilon^2\|\nabla^{k+2}\mathfrak{z}_\varepsilon^{-1}(a,\xi)\|_{L^p} \leqslant c\|(a,\xi)\|_{W^{k,p}}.$$

Proof of Proposition 2.7.6. It suffices to prove the result for $\mathbf{p} = \mathbf{p}_0$, for then it follows for $\mathbf{p}$ close to $\mathbf{p}_0$.

Recall that

$$\begin{aligned}\bar{X}_\varepsilon &= \operatorname{coker}(\mathrm{d}\mathfrak{F})_{s_0} \oplus W^{k+1,p}\Gamma(\mathfrak{H}) \\ &\quad \oplus W^{k+1,p}\Gamma(\mathfrak{N}) \\ &\quad \oplus W^{k+2,p}\Omega^1(M,\mathfrak{g}_P) \oplus W^{k+2,p}\Omega^0(M,\mathfrak{g}_P), \\ \bar{Y} &= I_\partial \oplus \mathbf{R} \oplus W^{k,p}\Gamma(\mathfrak{H}) \\ &\quad \oplus W^{k,p}\Gamma(\mathfrak{N}) \\ &\quad \oplus W^{k+1,p}\Omega^1(M,\mathfrak{g}_P) \oplus W^{k+1,p}\Omega^0(M,\mathfrak{g}_P),\end{aligned}$$

and $\bar{L}_{\mathbf{p}_0,\varepsilon}$ can be written as

$$\begin{pmatrix} -\bar{\not{D}}_{\mathfrak{H}} & \gamma II^* & 0 \\ -\gamma II & -\not{D}_{\mathfrak{N}} & -\mathfrak{a} \\ 0 & -\mathfrak{a}^* & \varepsilon^2\delta_{A_0} \end{pmatrix}$$

with

$$\delta_{A_0} = \begin{pmatrix} *\mathrm{d}_{A_0} & \mathrm{d}_{A_0} \\ \mathrm{d}_{A_0}^* & 0 \end{pmatrix}.$$

The operators

$$\bar{\not{D}}_{\mathfrak{H}} \colon \operatorname{coker}(\mathrm{d}\mathfrak{F})_{s_0} \oplus W^{k+1,p}\Gamma(\mathfrak{H}) \to I_\partial \oplus \mathbf{R} \oplus W^{k,p}\Gamma(\mathfrak{H})$$

and

$$\mathfrak{a}^* \colon W^{k+1,p}\Gamma(\mathfrak{N}) \to W^{k+1,p}\Omega^1(M,\mathfrak{g}_P) \oplus W^{k+1,p}\Omega^0(M,\mathfrak{g}_P)$$

are both invertible with uniformly bounded inverses, and by Proposition 2.7.8 the same holds for $\mathfrak{z}_\varepsilon$, provided $\varepsilon \ll 1$. Thus, according to Proposition 2.7.7, $\bar{L}_{\mathbf{p}_0,\varepsilon}$ has a left inverse $R_\varepsilon \colon \bar{Y}_0 \to \bar{X}_\varepsilon$ whose norm can be bounded independent of ε.

To see that R_ε is also a right inverse, observe that $L_{\mathbf{p}_0,\varepsilon}$ is a formally self-adjoint elliptic operator and, hence, $L_{\mathbf{p}_0,\varepsilon} \colon X_\varepsilon \to Y$ is Fredholm of index zero. Consequently, $\bar{L}_{\mathbf{p}_0,\varepsilon}$ is Fredholm of index zero. The existence of R_ε shows that $\ker \bar{L}_{\mathbf{p}_0,\varepsilon} = 0$ and thus $\operatorname{coker} \bar{L}_{\mathbf{p}_0,\varepsilon} = 0$. By the Open Mapping Theorem, $\bar{L}_{\mathbf{p}_0,\varepsilon}$ has an inverse $\bar{L}_{\mathbf{p}_0,\varepsilon}^{-1}$. It must agree with R_ε since $R_\varepsilon = R_\varepsilon \bar{L}_{\mathbf{p}_0,\varepsilon}\bar{L}_{\mathbf{p}_0,\varepsilon}^{-1} = \bar{L}_{\mathbf{p}_0,\varepsilon}^{-1}$. □

Proof of Proposition 2.7.7. The left inverse of L can be found by Gauss elimination, as in the Proof of Proposition 2.6.1. In fact, the exact formula for L is not needed; the point is that the only operators that we need to invert in the process of Gaussian elimination are exactly D_1, A, and Z. The calculation is elementary but unenlightening, and we refer an interested reader to [DW17a, Proof of Proposition 6.1]. □

Proof of Proposition 2.7.8. It suffices to show that the linear maps $\tilde{\mathfrak{z}}_\varepsilon := \mathfrak{a}^*\mathfrak{z}_\varepsilon$ are uniformly invertible. A short computation using Proposition 2.3.9 shows that

$$\tilde{\mathfrak{z}}_\varepsilon = \varepsilon^2\delta_{A_0}^2 + \mathfrak{a}^*\mathfrak{a} + \varepsilon^2\mathfrak{e}$$

where $\mathfrak{e}$ is a zeroth order operator which factors through $W^{k+1,p} \to W^{k+1,p}$. Since Φ_0 is regular, $\mathfrak{a}^*\mathfrak{a}$ is positive definite and, hence, for $\varepsilon \ll 1$, $\mathfrak{a}^*\mathfrak{a} + \varepsilon^2\delta_{A_0}^2$ is uniformly invertible. Since $\varepsilon \ll 1$, $\varepsilon^2\mathfrak{e}$ is a small perturbation of order ε and thus $\tilde{\mathfrak{z}}_\varepsilon$ is uniformly invertible. □

The above analysis yields the following regularity result, in which we decorate X_ε and Y with superscripts indicating the choice of the differentiability and integrability parameters k and p. The proof is almost identical to that of Proposition 2.6.5, and will be omitted.

Proposition 2.7.9. *For each $k, \ell \in \mathbf{N}$ and $p, q \in (1, \infty)$ with $(k+1)p > 3$, $\ell \geqslant k$, and $q \geqslant p$, there are constants $c, \sigma, \varepsilon_0 > 0$ and an open neighborhood U of $\mathbf{p}_0$ in $\mathscr{P}$ such that if $\varepsilon \in (0, \varepsilon_0]$, $\mathbf{p} \in U$, and $\hat{\mathfrak{c}} \in B_\sigma(0) \subset X_\varepsilon^{k,p}$ is solution of*

$$L_{\mathbf{p},\varepsilon}\hat{\mathfrak{c}} + Q_{\mathbf{p},\varepsilon}(\hat{\mathfrak{c}}) + \mathfrak{e}_{\mathbf{p},\varepsilon} = 0,$$

then $\hat{\mathfrak{c}} \in X_\varepsilon^{\ell,q}$ and $\|\hat{\mathfrak{c}}\|_{X_\varepsilon^{\ell,q}} \leqslant c\|\hat{\mathfrak{c}}\|_{X_\varepsilon^{k,p}}$.

2.7.3 *Proof of Theorem 2.7.1*

Since $Q_{\mathbf{p},\varepsilon}$ is quadratic and

$$\begin{aligned}\|Q_{\mathbf{p},\varepsilon}(\phi, a, \xi)\|_Y &\leqslant \|\bar{\gamma}(a)\phi\|_{W^{k,p}} + \varepsilon^2\|[a \wedge a]\|_{W^{k+1,p}} + \|\mu(\phi)\|_{W^{k+1,p}} + \|\phi\|_{L^2}^2 \\ &\lesssim \|a\|_{W^{k,p}}\|\phi\|_{W^{k+1,p}} \\ &\quad + \left(\|a\|_{W^{k,p}} + \varepsilon\|\nabla^{k+1}a\|_{L^p} + \varepsilon^2\|\nabla^{k+2}a\|_{L^p}\right)^2 + \|\phi\|_{W^{k+1,p}}^2,\end{aligned}$$

$Q_{\mathbf{p},\varepsilon}$ satisfies (2.3.4), and because of Proposition 2.7.6 we can apply Lemma 2.3.13 to construct a smooth map $\mathrm{ob}_\circ\colon U \times (0, \varepsilon_0) \times \mathscr{I}_\partial \to \operatorname{coker}(\mathrm{d}\mathfrak{F})_{s_0}$ and a map $\mathfrak{x}_\circ\colon \mathrm{ob}^{-1}(0) \to \overline{\mathfrak{M}}_{\mathrm{SW}}$ which is a homeomorphism onto the intersection of $\mathfrak{M}_{\mathrm{SW}}$ with a neighborhood of $[(A_0, \Phi_0)]$. (There is a slight caveat in the application of Lemma 2.3.13: the Banach space X_ε does depend on $\mathbf{p}$ and ε and Y depends on $\mathbf{p}$. The dependence, however, is mostly harmless as different values of $\mathbf{p}$ and ε lead to naturally isomorphic Banach spaces.) For what follows it will be important to know that maps $\mathrm{ob}_\circ$ and $\mathfrak{x}_\circ$ are uniquely characterized as follows: for $\mathbf{p}$ in the open neighborhood U of $\mathbf{p}_0 \in \mathscr{P}$, d in the open neighborhood $\mathscr{I}_\partial$ of $0 \in I_\partial$, and $\varepsilon \in (0, \varepsilon_0)$, there is a unique solution $\bar{\mathfrak{c}} = \bar{\mathfrak{c}}(\mathbf{p}, \varepsilon, d) \in B_\sigma(0) \subset \bar{X}_\varepsilon$ of

$$\bar{L}_{\mathbf{p},\varepsilon}\bar{\mathfrak{c}} + Q_{\mathbf{p},\varepsilon}(\bar{\mathfrak{c}}) + \mathfrak{e}_{\mathbf{p},\varepsilon} = d \in \mathscr{I}_\partial \subset \bar{Y}; \tag{2.7.6}$$

$\mathrm{ob}_\circ(\mathbf{p}, \varepsilon, d)$ is the component of $\bar{\mathfrak{c}}(\mathbf{p}, \varepsilon, d)$ in $\operatorname{coker}(\mathrm{d}\mathfrak{F})_{s_0}$ and if $\mathrm{ob}_\circ(\mathbf{p}, \varepsilon, d) = 0$ and $\hat{\mathfrak{c}}$ denotes the component of $\mathfrak{c}(\mathbf{p}, \varepsilon, d)$ in X_ε, then $\mathfrak{x}_\circ(\mathbf{p}, \varepsilon, d) = \mathfrak{c}_0 + \hat{\mathfrak{c}}$. (Similar, setting $\varepsilon = 0$ yields ob_∂ and $\mathfrak{x}_\partial$.)

We define $\mathrm{ob}\colon U \times [0, \varepsilon_0) \times \mathscr{I}_\partial \to \operatorname{coker}(\mathrm{d}\mathfrak{F})_{s_0}$ by

$$\mathrm{ob}(\cdot, \varepsilon, \cdot) = \begin{cases} \mathrm{ob}_\circ(\cdot, \varepsilon, \cdot) & \text{for } \varepsilon \in (0, \varepsilon_0) \\ \mathrm{ob}_\partial(\cdot, \cdot) & \text{for } \varepsilon = 0, \end{cases}$$

and $\mathfrak{x}\colon \mathrm{ob}^{-1}(0) \to \overline{\mathfrak{M}}_{\mathrm{SW}}$ by

$$\mathfrak{x}(\cdot, \varepsilon, \cdot) = \begin{cases} \mathfrak{x}_\circ(\cdot, \varepsilon, \cdot) & \text{for } \varepsilon \in (0, \varepsilon_0) \\ \mathfrak{x}_\partial(\cdot, \cdot) & \text{for } \varepsilon = 0. \end{cases}$$

In order to prove Theorem 2.7.1 we need to understand the regularity of ob near $\varepsilon = 0$; in other words: we need to understand how $\mathrm{ob}_\circ$ and ob_∂ fit together.

Let $k \in \mathbf{N}$ and $p \in (1,\infty)$ be the differentiability and integrability parameters used in the definition of $\bar{X}_\varepsilon$. If necessary, shrink U and $\mathscr{I}_\partial$ and decrease σ so that the proof of Proposition 2.6.1 goes through and Proposition 2.6.2 holds with differentiability parameter $k+2r+2$ and integrability parameter p. Observe that $\bar{X}_0^{k+2,p} \subset \bar{X}_\varepsilon$ and the norm of the inclusion can be bounded by a constant independent of ε.

Proposition 2.7.10. *For every $(\mathbf{p},d) \in U \times \mathscr{I}_\partial$, there are $\bar{\mathfrak{c}}_0(\mathbf{p},d) \in \bar{X}_0^{k+2r+2,p}$ and $\hat{\mathfrak{c}}_i(\mathbf{p},d) \in \bar{X}_0^{k+2(r-i)+2,p}$ (for $i = 1,\dots,r$) depending smoothly on $\mathbf{p}$ and d, such that, for $m,n \in \mathbf{N}$ with $m+n \leqslant 2r$,*

$$\tilde{\mathfrak{c}}(\mathbf{p},\varepsilon,d) := \bar{\mathfrak{c}}_0 + \sum_{i=1}^{r} \varepsilon^{2i}\hat{\mathfrak{c}}_i$$

satisfies

$$\left\| \nabla^m_{U\times\mathscr{I}_\partial}\partial^n_\varepsilon\big(\bar{\mathfrak{c}}(\mathbf{p},\varepsilon,d) - \tilde{\mathfrak{c}}(\mathbf{p},\varepsilon,d)\big)\right\|_{\bar{X}_\varepsilon} = O(\varepsilon^{2k+2-n}). \tag{2.7.7}$$

Proof. We construct $\tilde{\mathfrak{c}}$ by expanding (2.7.6) in ε^2. To this end we write

$$\bar{L}_{\mathbf{p},\varepsilon} = \bar{L}_{\mathbf{p},0} + \varepsilon^2\ell_{\mathbf{p}}, \quad Q_{\mathbf{p},\varepsilon}, = Q_{\mathbf{p},0} + \varepsilon^2 q_{\mathbf{p}}, \quad \text{and} \quad \mathfrak{e}_{\mathbf{p},\varepsilon} = \mathfrak{e}_{\mathbf{p},0} + \varepsilon^2\hat{\mathfrak{e}}_{\mathbf{p}},$$

with

$$\ell_{\mathbf{p}} := \begin{pmatrix} 0 & & \\ & 0 & \\ & & \delta_{A_0}\end{pmatrix}, \quad q_{\mathbf{p}}(\phi,a,\xi) := \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2}*[a\wedge a]\end{pmatrix}, \quad \text{and} \quad \hat{\mathfrak{e}}_{\mathbf{p}} := \begin{pmatrix} 0 \\ 0 \\ *\varpi F_{A_0}\end{pmatrix}.$$

Observe that $\ell_{\mathbf{p}}\colon \bar{X}_0^{\ell,p} \to \bar{Y}^{\ell-2,p}$ is a bounded linear map and $q_{\mathbf{p}}\colon \bar{X}_0^{\ell,p} \to \bar{Y}^{\ell-2,p}$ is a bounded quadratic map.

Step 1. *Construction of $\bar{\mathfrak{c}}_0$ and $\hat{\mathfrak{c}}_i$.*

By Banach's Fixed Point Theorem, there is a unique solution $\bar{\mathfrak{c}}_0 \in B_\sigma(0) \subset \bar{X}_0^{k+2r+2,p}$ of

$$\bar{L}_{\mathbf{p},0}\bar{\mathfrak{c}}_0 + Q_{\mathbf{p},0}(\bar{\mathfrak{c}}_0) + \mathfrak{e}_{\mathbf{p},0} = d \in \mathscr{I}_\partial \subset \bar{Y}^{k+2r+2}.$$

Moreover, $\bar{\mathfrak{c}}_0$ actually lies in $B_{\sigma/2}(0) \subset \bar{X}_0^{k+2r+2,p}$ provided U and $\mathscr{I}_\partial$ have been chosen sufficiently small. We have

$$\bar{L}_{\mathbf{p},\varepsilon}\bar{\mathfrak{c}}_0 + Q_{\mathbf{p},0}(\bar{\mathfrak{c}}_\varepsilon) + \mathfrak{e}_{\mathbf{p},\varepsilon} - d = \varepsilon^2\mathfrak{r}_0(\mathbf{p},d) \in \bar{Y}^{k+2(r-1)+2,p}.$$

with

$$\mathfrak{r}_0(\mathbf{p},d) := \ell_{\mathbf{p}}\bar{\mathfrak{c}}_0 + q_{\mathbf{p}}(\bar{\mathfrak{c}}_0) + \hat{\mathfrak{e}}_{\mathbf{p}}.$$

Since $\sigma \ll 1$, the operator $\bar{L}_{\mathbf{p},0} + 2Q_{\mathbf{p},0}(\bar{\mathfrak{c}}_0,\cdot)\colon \bar{X}_0^{k+2(r-i)+2,p} \to \bar{Y}_0^{k+2(r-i)+2,p}$ is invertible for $i = 1,\dots,r$.[5] Recursively define $\mathfrak{r}_i(\mathbf{p},d) \in \bar{Y}^{k+2(r-i-1)+2,p}$ by

$$\varepsilon^{2i+2}\mathfrak{r}_i := \bar{L}_{\mathbf{p},\varepsilon}\bar{\mathfrak{c}}^i_\varepsilon + Q_{\mathbf{p},0}(\bar{\mathfrak{c}}^i_\varepsilon) + \mathfrak{e}_{\mathbf{p},\varepsilon} - d$$

5 Here we engage in the slight abuse of notation to use the same notation for a bilinear map and its associated quadratic form.

with

$$\tilde{\mathfrak{c}}(\varepsilon, \mathbf{p}, d) := \bar{\mathfrak{c}}_0 + \varepsilon^2 \hat{\mathfrak{c}}_1 + \cdots + \varepsilon^{2i} \hat{\mathfrak{c}}_i,$$

and define $\hat{\mathfrak{c}}_{i+1} \in \bar{X}_0^{k+2(r-i-1)+2}$ to be the unique solution of

$$\bar{L}_{\mathbf{p},0} \hat{\mathfrak{c}}_{i+1} + 2Q_{\mathbf{p},0}(\bar{\mathfrak{c}}_0, \hat{\mathfrak{c}}_{i+1}) = \mathfrak{r}_i.$$

Clearly, $\bar{\mathfrak{c}}_0, \hat{\mathfrak{c}}_1, \ldots, \hat{\mathfrak{c}}_r$ depend smoothly on $\mathbf{p}$ and d.

Step 2. *We prove* (2.7.7).

We have

$$\bar{L}_{\mathbf{p},\varepsilon} \bar{\mathfrak{c}}_\varepsilon + Q_{\mathbf{p},\varepsilon}(\bar{\mathfrak{c}}_\varepsilon) - \bar{L}_{\mathbf{p},\varepsilon} \tilde{\mathfrak{c}} - Q_{\mathbf{p},\varepsilon}(\tilde{\mathfrak{c}}) = -\varepsilon^{2k+2} \mathfrak{r} \tag{2.7.8}$$

with $\mathfrak{r} = \mathfrak{r}_r$ as in the previous step. Both $\bar{\mathfrak{c}}$ and $\tilde{\mathfrak{c}}$ are small in $\bar{X}_\varepsilon$; hence, it follows that

$$\|\bar{\mathfrak{c}} - \tilde{\mathfrak{c}}\|_{\bar{X}_\varepsilon} = O(\varepsilon^{2k+2}).$$

To obtain estimates for the derivatives of $\bar{\mathfrak{c}} - \tilde{\mathfrak{c}}$, we differentiate (2.7.8) and obtain an identity whose left-hand side is

$$\bar{L}_{\mathbf{p},0} \nabla^m \partial_\varepsilon^n (\bar{\mathfrak{c}} - \tilde{\mathfrak{c}}) + 2Q_{\mathbf{p},0}\left(\bar{\mathfrak{c}}, \nabla^m \partial_\varepsilon^n (\bar{\mathfrak{c}} - \tilde{\mathfrak{c}})\right) + 2Q_{\mathbf{p},0}\left(\bar{\mathfrak{c}} - \tilde{\mathfrak{c}}, \nabla^m \partial_\varepsilon^n \tilde{\mathfrak{c}}\right)$$

and whose right-hand side can be controlled in terms of the lower order derivatives of $\hat{\mathfrak{d}}_\varepsilon^k$. This gives the asserted estimates. □

From Proposition 2.7.10 it follows that $\mathfrak{x}$ is a homeomorphism onto its image and that the estimate in Theorem 2.7.1(1) holds with $\widehat{\mathrm{ob}}_i$ denoting the component of $\hat{\mathfrak{c}}_i$ in $\mathrm{coker}(\mathrm{d}\mathfrak{F})_{s_0}$. This expansion implies that ob is C^{2r-1} up to $\varepsilon = 0$. □

2.8 A WALL-CROSSING PHENOMENON

The main application of Theorem 2.7.1—and our motivation for proving it—is to understand wall-crossing phenomena for signed counts of solutions to Seiberg–Witten equations arising from the non-compactness phenomenon mentioned in Section 2.5. This phenomenon, for one particular example of a Seiberg–Witten equation, will be studied in greater detail in the next chapter, and the result described below will play a crucial role in this study.

In the generic situation of Theorem 2.7.1, one expects to have $\ker(\mathrm{d}\mathfrak{F})_{s_0} = \mathbf{R}\langle \hat{v} \circ s_0 \rangle$. In this case, if $\{\mathbf{p}_t = (g_t, B_t) : t \in (-T, T)\}$ is a 1–parameter family in $\mathscr{P}$, then (for $T \ll 1$) one can find a 1–parameter family $\{(s_t) \in \Gamma(\mathfrak{X}) : t \in (-T, T)\}$ of sections of $\mathfrak{X}$ and $\lambda\colon (-T, T) \to \mathbf{R}$ with $\lambda(0) = 0$ such that

$$\mathfrak{F}_t(s_t) = \lambda(t) \cdot \hat{v} \circ s_t.$$

Theorem 2.8.1. *In the situation above and assuming $\dot{\lambda}(0) \neq 0$, for each $r \in \mathbf{N}$, there exist $\varepsilon_0 > 0$ and C^{2r-1} maps $t\colon [0, \varepsilon_0) \to (-T, T)$ and $\mathfrak{c}\colon [0, \varepsilon_0) \to \Gamma(\mathfrak{S}^{\mathrm{reg}}) \times \mathscr{A}(Q)$ such that an open neighborhood V of $(0, 0, [\mathfrak{c}_0])$ in the parametrized Seiberg–Witten moduli space*

$$\left\{ (t, \varepsilon, [(\Phi, A)]) \in (-T, T) \times [0, \infty) \times \frac{\Gamma(\mathfrak{S}) \times \mathscr{A}(Q)}{\mathscr{G}(P)} : (\varepsilon, [(\Phi, A)]) \in \overline{\mathfrak{M}}_{\mathrm{SW}}(\mathbf{p}_t) \right\}$$

is given by

$$V = \{(t(\varepsilon), \varepsilon, [\mathfrak{c}(\varepsilon)]) : \varepsilon \in [0, \varepsilon_0)\}.$$

If $\mathfrak{c}(\varepsilon) = (\Phi(\varepsilon), A(\varepsilon))$*, then there is* $\phi \in \Gamma(\mathfrak{S})$ *such that*

$$\Phi(\varepsilon) = \Phi_0 + \varepsilon^2\phi + O(\varepsilon^4),$$

and with

$$\delta := \langle \phi, \not{D}_{A_0}\phi \rangle_{L^2}$$

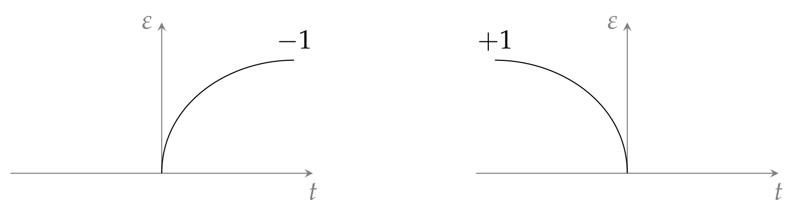
we have

$$t(\varepsilon) = \frac{\delta}{\dot{\lambda}(0)}\varepsilon^4 + O(\varepsilon^6).$$

For $\varepsilon \in (0, \varepsilon_0)$*,* $\mathfrak{c}(\varepsilon)$ *is irreducible; moreover, if* $\delta \neq 0$*, then* $\mathfrak{c}(\varepsilon)$ *is unobstructed.*

Remark 2.8.2. In the situation of Theorem 2.8.1, there is no obstruction to solving the Seiberg–Witten equation to order ε^2—in fact, a solution can be found rather explicitly. The obstruction to solving to order ε^4 is precisely δ.

If $\mathfrak{M}_{\mathrm{SW}}$ is oriented (that is: $\det L \to \mathfrak{M}_{\mathrm{SW}}$ is trivialized) around $(\mathbf{p}_0, [\mathfrak{c}_0])$, then identifying $\ker(\mathrm{d}\mathfrak{F})_{s_0} = \operatorname{coker}(\mathrm{d}\mathfrak{F})_{s_0} = \mathbf{R}\langle \hat{v} \circ s \rangle$ determines a sign $\sigma = \pm 1$. If $\delta \neq 0$, then contribution of $[\mathfrak{c}(\varepsilon)]$ should be counted with sign $-\sigma \cdot \operatorname{sign}(\delta)$; as is discussed in Section 2.4. However, $\operatorname{sign}(\delta/\dot{\lambda}(0))$ also determines whether the solution $\mathfrak{c}(\varepsilon)$ appears for $t < 0$ or $t > 0$. Thus, the overall contributions from $\operatorname{sign}(\delta)$ cancel.



(a) $\dot{\lambda}(0) > 0, \sigma = +1, \delta > 0$ (b) $\dot{\lambda}(0) > 0, \sigma = +1, \delta < 0$

Figure 2.1: Two examples of wall-crossing.

This is illustrated in Figure 2.1, which depicts two examples of wall-crossing. More precisely, it shows the projection of $\bigcup_{t \in (-T,T)} \overline{\mathfrak{M}}_{\mathrm{SW}}(\mathbf{p}_t)$ on the (t, ε)-plane. In both cases we assume $\dot{\lambda}(0) > 0$ and $\sigma = +1$. Figure 2.1a represents the case $\delta > 0$, in which a solution $\mathfrak{c}(\varepsilon)$ with sign $\operatorname{sign}(\mathfrak{c}(\varepsilon)) = -\sigma \cdot \operatorname{sign}(\delta) = -1$ is born at $t = 0$. Figure 2.1b represents the case $\delta < 0$, in which $\operatorname{sign}(\mathfrak{c}(\varepsilon)) = +1$ and the solution dies at $t = 0$. In both cases, as we cross from $t < 0$ to $t > 0$ the signed count of solutions to the Seiberg–Witten equation changes by -1.

Proof of Theorem 2.8.1. The first part of Theorem 2.8.1 follows directly from Theorem 2.7.1, since in this situation

$$\operatorname{ob}(\varepsilon, t) = \dot{\lambda}(0) \cdot t + O(t^2) + O(\varepsilon^2)$$

because $\mathrm{ob}_\partial(t) = \dot\lambda(0)\cdot t + O(t^2)$. The second part requires a more detailed analysis to show that

$$\mathrm{ob}(\varepsilon,t) = \dot\lambda(0)\cdot t - \delta\varepsilon^4 + O(t^2) + O(\varepsilon^6).$$

To establish the above expansion of ob, we solve

$$\bar L_\varepsilon(o_\varepsilon,\hat{\mathfrak{c}}) + Q_\varepsilon(\hat{\mathfrak{c}}) + \begin{pmatrix} 0 \\ 0 \\ \varepsilon^2 * \varpi F_{A_0} \\ 0 \end{pmatrix} = 0$$

by formally expanding in ε^2. Inspection of (2.6.5) shows that the obstruction to being able to solve $L_0\hat{\mathfrak{c}} = (\psi,b,\eta)$ is

$$-\pi\big(\psi + \gamma II(\mathfrak{a}^*)^{-1}(b,\eta)\big)$$

where π denotes the L^2–orthogonal projection onto $\ker \not{D}_{\mathfrak{H}}$. In the case at hand, $\ker \not{D}_{\mathfrak{H}} = \mathbf{R}\langle\Phi_0\rangle$, and we have

$$\begin{aligned}\langle\Phi_0,\gamma II^*(\mathfrak{a}^*)^{-1}(b,\eta)\rangle_{L^2} &= \sum_{i=1}^3 \langle\Phi_0,\gamma(e_i)\nabla_{e_i}(\mathfrak{a}^*)^{-1}(b,\eta)\rangle_{L^2} \\ &= \sum_{i=1}^3 \langle\gamma(e_i)\nabla_{e_i}\Phi_0,(\mathfrak{a}^*)^{-1}(b,\eta)\rangle_{L^2} = 0\end{aligned}$$

since $\mathfrak{a}\colon \Omega^1(M,\mathfrak{g}_P)\oplus\Omega^0(M,\mathfrak{g}_P) \to \Gamma(\mathfrak{N})$ and thus $(\mathfrak{a}^*)^{-1}$ also maps to $\Gamma(\mathfrak{N})$. Thus the obstruction reduces to

$$-\langle\Phi_0,\psi\rangle_{L^2}.$$

By (2.6.5), the solution to $L_0(\phi,a,\xi) = (0,*\varpi F_{A_0},0)$ is

$$\begin{aligned}\phi &= -\not{D}_{\mathfrak{H}}^{-1}\gamma II^*\chi - \chi, \quad \text{and} \\ (a,\xi) &= \mathfrak{a}^{-1}\not{D}_{\mathfrak{N}}\chi + \mathfrak{a}^{-1}\gamma II\not{D}_{\mathfrak{H}}^{-1}\gamma II^*\chi\end{aligned} \tag{2.8.1}$$

with

$$\chi := (\mathfrak{a}^*)^{-1} * \varpi F_{A_0}. \tag{2.8.2}$$

Setting $\hat{\mathfrak{c}}_0 := \varepsilon^2(\phi,a,\xi)$, we have

$$\varepsilon^4\hat{\mathfrak{d}}_1 := \bar L_\varepsilon(0,\hat{\mathfrak{c}}_0) + Q_\varepsilon(\hat{\mathfrak{c}}_0) + (0,0,\varepsilon^2 * \varpi F_{A_0},0) = O(\varepsilon^4).$$

The component of $\hat{\mathfrak{d}}_1$ in $\Gamma(\mathfrak{S})$ is

$$-\bar\gamma(a)\phi.$$

Using $\bar\gamma(a)\Phi_0 \in \Gamma(\mathfrak{N})$ and $\rho(\mathfrak{g}_P)\Phi \perp \chi$, we find that the obstruction to being able to solve $L_0(\phi_1,a_1,\xi_1) = \hat{\mathfrak{d}}_1$ is

$$\begin{aligned}\mathfrak{o} := \langle\Phi_0,\bar\gamma(a)\phi\rangle_{L^2} &= \langle\bar\gamma(a)\Phi_0,\phi\rangle_{L^2} \\ &= -\langle\bar\gamma(a)\Phi_0,\chi\rangle_{L^2} \\ &= -\langle\mathfrak{a}(a,\xi),\chi\rangle_{L^2} \\ &= -\langle\not{D}_{\mathfrak{N}}\chi + \gamma II\not{D}_{\mathfrak{H}}^{-1}\gamma II^*\chi,\chi\rangle_{L^2} \\ &= -\langle\not{D}_{\mathfrak{N}}\chi,\chi\rangle_{L^2} + \langle\not{D}_{\mathfrak{H}}^{-1}\gamma II^*\chi,\gamma II^*\chi\rangle_{L^2}.\end{aligned}$$

Comparing this with

$$\begin{aligned}
\langle \not{D}_{A_0}\phi, \phi\rangle_{L^2} &= \langle \not{D}_{A_0}\not{D}_{\mathfrak{H}}^{-1}\gamma \mathrm{II}^*\chi + \not{D}_{A_0}\chi, \not{D}_{\mathfrak{H}}^{-1}\gamma \mathrm{II}^*\chi + \chi\rangle_{L^2} \\
&= \langle (\not{D}_{\mathfrak{H}} + \gamma \mathrm{II})\not{D}_{\mathfrak{H}}^{-1}\gamma \mathrm{II}^*\chi + (\not{D}_{\mathfrak{N}} - \gamma \mathrm{II}^*)\chi, \not{D}_{\mathfrak{H}}^{-1}\gamma \mathrm{II}^*\chi + \chi\rangle_{L^2} \\
&= \langle \gamma \mathrm{II}^*\chi, \not{D}_{\mathfrak{H}}^{-1}\gamma \mathrm{II}^*\chi\rangle_{L^2} + \langle \gamma \mathrm{II}\not{D}_{\mathfrak{H}}^{-1}\gamma \mathrm{II}^*, \chi\rangle_{L^2} \\
&\quad + \langle \not{D}_{\mathfrak{N}}\chi, \chi\rangle_{L^2} - \langle \gamma \mathrm{II}^*\chi, \not{D}_{\mathfrak{H}}^{-1}\gamma \mathrm{II}^*\chi\rangle_{L^2} \\
&= -\langle \not{D}_{\mathfrak{H}}^{-1}\gamma \mathrm{II}^*, \gamma \mathrm{II}^*\chi\rangle_{L^2} + \langle \not{D}_{\mathfrak{N}}\chi, \chi\rangle_{L^2} \\
&= -0
\end{aligned}$$

completes the proof. □

2.9 ADHM MONOPOLES

The main example of a Seiberg–Witten equation that we will consider is the *ADHM*$_{r,k}$ *Seiberg–Witten equation*. According to the Haydys–Walpuski program described in the introduction, this equation should play an important role in gauge theory on G_2–manifolds. More precisely, we expect solutions of the ADHM$_{r,k}$ Seiberg–Witten equation to play a role in counter-acting the bubbling phenomenon along associative submanifolds discussed in [DS11; Wal17]. This equation arises from the general construction discussed in Section 2.2 for the following quaternionic representation.

Example 2.9.1. The *ADHM*$_{r,k}$ *representation* is given by the quaternionic vector space

$$S = \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^r, \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k) \oplus \mathbf{H}^* \otimes_{\mathbf{R}} \mathfrak{u}(k)$$

with a quaternionic action of

$$G = \mathrm{U}(k) \triangleleft H = \mathrm{SU}(r) \times \mathrm{Sp}(1) \times \mathrm{U}(k).$$

The group $\mathrm{SU}(r)$ acts on $\mathbf{C}^r$ in the obvious way, $\mathrm{U}(k)$ acts on $\mathbf{C}^k$ in the obvious way and on $\mathfrak{u}(k)$ by the adjoint representation, and $\mathrm{Sp}(1)$ acts on the first copy of $\mathbf{H}$ trivially and on the second copy by right-multiplication with the conjugate.

According to Atiyah et al. [Ati+78], if $r \geqslant 2$, then the hyperkähler quotient $S^{\mathrm{reg}} /\!/\!/ G$ is the moduli space of framed $\mathrm{SU}(r)$ ASD instantons of charge k on $\mathbf{R}^4$, and $\mu^{-1}(0)/G$ is its Uhlenbeck compactification. If $r = 1$, then $\mu^{-1}(0) \cap S^{\mathrm{reg}} = \varnothing$, and $\mu^{-1}(0)/G = \mathrm{Sym}^k \mathbf{H} := \mathbf{H}^k/S_k$ by Nakajima [Nak99, Example 3.14].

Let us write down the equations explicitly.

Remark 2.9.2. Below we consider a slight generalization of the construction from Section 2.2, allowing for a choice of a so-called $\mathrm{spin}^{\mathrm{U}(k)}$ structure rather than a spin structure together with a $\mathrm{U}(k)$–bundle, cf. Remark 2.2.2. Thus, equation (2.9.3), introduced below, is more general than the Seiberg–Witten equation (2.2.1) associated with the ADHM$_{r,k}$ representation.

Definition 2.9.3. Let M be an oriented Riemannian 3–manifold. Consider the Lie group

$$\mathrm{Spin}^{\mathrm{U}(k)}(n) := (\mathrm{Spin}(n) \times \mathrm{U}(k))/\mathbf{Z}_2.$$

A *spin*$^{\mathrm{U}(k)}$ *structure* on M is a principal $\mathrm{Spin}^{\mathrm{U}(k)}(3)$–bundle together with an isomorphism

$$\mathfrak{w} \times_{\mathrm{Spin}^{\mathrm{U}(k)}(3)} \mathrm{SO}(3) \cong \mathrm{SO}(TM). \tag{2.9.1}$$

The *spinor bundle* and the *adjoint bundle* associated with a $\mathrm{spin}^{\mathrm{U}(k)}$ structure $\mathfrak{w}$ are

$$W := \mathfrak{w} \times_{\mathrm{Spin}^{\mathrm{U}(k)}(3)} \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^k \quad \text{and} \quad \mathfrak{g}_{\mathscr{H}} := \mathfrak{w} \times_{\mathrm{Spin}^{\mathrm{U}(k)}(3)} \mathfrak{u}(k)$$

respectively. The left multiplication by $\operatorname{Im}\mathbf{H}$ on $\mathbf{H} \otimes \mathbf{C}^k$ induces a *Clifford multiplication* $\gamma\colon TM \to \operatorname{End}(W)$.

A *spin connection* on $\mathfrak{w}$ is a connection A inducing the Levi-Civita connection on TM. Associated with each spin connection A there is a *Dirac operator* $\not{D}_A\colon \Gamma(W) \to \Gamma(W)$.

Denote by $\mathscr{A}^s(\mathfrak{w})$ the space of spin connections on $\mathfrak{w}$, and by $\mathscr{G}^s(\mathfrak{w})$ the *restricted gauge group*, consisting of those automorphisms of $\mathfrak{w}$ which act trivially on TM. Let $\varpi\colon \operatorname{Ad}(\mathfrak{w}) \to \mathfrak{g}_{\mathscr{H}}$ be the map induced by the projection $\mathfrak{spin}^{\mathrm{U}(k)}(3) \to \mathfrak{u}(k)$.

Example 2.9.4. If $\mathfrak{s}$ is a spin structure on M with spinor bundle $\$$ and $\mathscr{H}$ is a rank k Hermitian vector bundle, then the principal $\mathrm{Spin}(n) \times \mathrm{U}(k)$–bundle $\mathfrak{s} \times \mathrm{Fr}(\mathscr{H})$ induces naturally a $\mathrm{spin}^{\mathrm{U}(k)}$ structure $\mathfrak{w}$ on M (here $\mathrm{Fr}(\mathscr{H})$ is the frame bundle of $\mathscr{H}$). The corresponding spinor bundle and adjoint bundle are, respectively:

$$W = \$ \otimes_{\mathbf{C}} \mathscr{H} \quad \text{and} \quad \mathfrak{g}_{\mathscr{H}} = \mathfrak{u}(\mathscr{H}).$$

Any $\mathrm{U}(k)$–connection on $\mathscr{H}$ gives rise to a connection in $\mathscr{A}^s(\mathfrak{w})$ and, conversely, every element of $\mathscr{A}^s(\mathfrak{w})$ arises in this way. The restricted gauge group $\mathscr{G}^s(\mathfrak{w})$ can be identified with the gauge group $\mathscr{G}(\mathscr{H})$ of $\mathrm{U}(k)$–gauge transformations.

If M is spin (e.g. if M is an oriented 3–manifold), then every $\mathrm{spin}^{\mathrm{U}(k)}$ structure arises in the way just described, although not uniquely.

Definition 2.9.5. Let M be an oriented 3–manifold. The *geometric data* needed to formulate the $\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation are:

- a Riemannian metric g,
- a $\mathrm{spin}^{\mathrm{U}(k)}$ structure $\mathfrak{w}$ with the corresponding spinor bundle W,
- a Hermitian vector bundle E of rank r with a fixed trivialization $\Lambda^r E = \mathbf{C}$ and an $\mathrm{SU}(r)$–connection B,
- an oriented Euclidean vector bundle V of rank 4 together with an isomorphism

  $$\mathrm{SO}(\Lambda^+ V) \cong \mathrm{SO}(TM) \tag{2.9.2}$$

  and an $\mathrm{SO}(4)$–connection C on V with respect to which this isomorphism is parallel.

Remark 2.9.6. If Y is a G_2–manifold and $M \subset Y$ an associative submanifold, then its normal bundle $V = N_{M/Y}$ admits a natural isomorphism (2.9.2) and we can take C to be the connection induced by the Levi-Civita connection. In this context, the bundle E should be the restriction to M of a bundle on the ambient G_2–manifold and B should be the restriction of a G_2–instanton. Observe that when $r = 1$, the bundle E and the connection B are necessarily trivial.

The above data makes both $\mathrm{Hom}(E, W)$ and $V \otimes \mathfrak{g}_{\mathscr{H}}$ into Clifford bundles over M; hence, there are Dirac operators

$$\not{D}_{A,B} \colon \Gamma(\mathrm{Hom}(E, W)) \to \Gamma(\mathrm{Hom}(E, W))$$

and

$$\not{D}_{A,C} \colon \Gamma(V \otimes \mathfrak{g}) \to \Gamma(V \otimes \mathfrak{g}).$$

The $\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation involves also two quadratic moment maps defined as follows. If $\Psi \in \mathrm{Hom}(E, W)$, then $\Psi\Psi^* \in \mathrm{End}(W)$. Since $\Lambda^2 T^*M \otimes \mathfrak{g}_{\mathscr{H}}$ acts on W, there is an adjoint map $(\cdot)_0 \colon \mathrm{End}(W) \to \Lambda^2 T^*M \otimes \mathfrak{g}_{\mathscr{H}}$. Define $\mu \colon \mathrm{Hom}(E, W) \to \Lambda^2 T^*M \otimes \mathfrak{g}_{\mathscr{H}}$ by

$$\mu(\Psi) := (\Psi\Psi^*)_0.$$

If $\xi \in V \otimes \mathfrak{g}$, then $[\xi \wedge \xi] \in \Lambda^2 V \otimes \mathfrak{g}_{\mathscr{H}}$. Denote its projection to $\Lambda^+ V \otimes \mathfrak{g}_{\mathscr{H}}$ by $[\xi \wedge \xi]^+$. Identifying $\Lambda^+ V \cong \Lambda^2 T^*M$ via the isomorphism (2.9.2), we define $\mu \colon V \otimes \mathfrak{g} \to \Lambda^2 T^*M \otimes \mathfrak{g}_{\mathscr{H}}$ by

$$\mu(\xi) := [\xi \wedge \xi]^+$$

Definition 2.9.7. Given a choice of geometric data as in Definition 2.9.5, the *$\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation* is the following partial differential equation for $(\Psi, \xi, A) \in \Gamma(\mathrm{Hom}(E, W)) \times \Gamma(V \otimes \mathfrak{g}_{\mathscr{H}}) \times \mathscr{A}^s(\mathfrak{w})$:

$$\begin{aligned} \not{D}_{A,B}\Psi &= 0, \\ \not{D}_{A,C}\xi &= 0, \quad \text{and} \\ \varpi F_A &= \mu(\Psi) + \mu(\xi). \end{aligned} \tag{2.9.3}$$

A solution of this equation is called an *$\mathrm{ADHM}_{r,k}$ monopole*.

The moduli space of $\mathrm{ADHM}_{r,k}$ monopoles might be non-compact. As in the previous section, to analyze this phenomenon, one considers the corresponding limiting equation, that is the $\varepsilon = 0$ version of (2.5.1)

Definition 2.9.8. The *limiting $\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation* the following partial differential equation for $(\Psi, \xi, A) \in \Gamma(\mathrm{Hom}(E, W)) \times \Gamma(V \otimes \mathfrak{g}_{\mathscr{H}}) \times \mathscr{A}^s(\mathfrak{w})$

$$\begin{aligned} \not{D}_{A,B}\Psi &= 0, \\ \not{D}_{A,C}\xi &= 0, \quad \text{and} \\ \mu(\Psi) + \mu(\xi) &= 0. \end{aligned} \tag{2.9.4}$$

together with the normalization $\|(\Psi, \xi)\|_{L^2} = 1$.

The $\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation (2.9.3) and the corresponding limiting equation are preserved by the action of the restricted gauge group $\mathscr{G}^s(\mathfrak{w})$.

Example 2.9.9. Suppose that $r = k = 1$. A $\mathrm{spin}^{\mathrm{U}(1)}$ structure is simply a spin^c structure and

$$\varpi F_A = \frac{1}{2} F_{\det A}.$$

Also, $\mathfrak{g}_{\mathscr{H}} = i\underline{\mathbf{R}}$; hence, $\not{D}_{A,C}$ is independent of A and $\mu(\xi) = 0$. The $\mathrm{ADHM}_{1,1}$ Seiberg–Witten equation is thus simply

$$\not{D}_A \Psi = 0 \quad \text{and} \quad \frac{1}{2} F_{\det A} = \mu(\Psi),$$

the classical Seiberg–Witten equation for (Ψ, A), together with the Dirac equation

$$\not{D}_C \xi = 0.$$

If $M \subset Y$ is an associative submanifold in a G_2–manifold and $V = N_{M/Y}$ its normal bundle, then $\not{D}_C$ is the Fueter operator, which controls the deformation theory of M; see [DW17b, Definition 2.18] In particular, ξ must vanish if M is an unobstructed associative submanifold.

(There is a variant of (2.9.3) in which ξ is taken to be a section of $V \otimes \mathfrak{g}^\circ_{\mathscr{H}}$ with $\mathfrak{g}^\circ_{\mathscr{H}}$ denoting the trace-free component of $\mathfrak{g}_{\mathscr{H}}$. For $r = k = 1$, this equation is identical to the classical Seiberg–Witten equation. However, from the viewpoint of G_2–geometry, it is more natural to consider (2.9.3).)

For larger values of r and k, little is currently known about the moduli spaces of $\mathrm{ADHM}_{r,k}$ monopoles and their compactifications. For $r = 1$, the equations resemble in many ways the equation for flat $G^{\mathbf{C}}$ connections from Example 2.2.11, studied extensively by Taubes [Tau13b]. Based on his work, and on a generalization of the Haydys correspondence, Walpuski and the author of this thesis proposed a conjecture describing limits of $\mathrm{ADHM}_{1,k}$ monopoles [DW17b, Section 5.4]. When $r = 1$, there is no background bundle E and connection B and, according to Remark 2.9.6, the $\mathrm{ADHM}_{1,k}$ equation is expected to play a role in constructing an enumerative theory based associative submanifolds (without G_2–instantons). The purpose of the article [DW17b] is to develop formal aspects of such a theory.

In this thesis, we focus on the equations corresponding to $r > 1$ and $k = 1$. As in Example 2.9.9, in this case W is the spinor bundle of a spin^c structure and the equation decouples into

$$\not{D}_{A,B} \Psi = 0 \quad \text{and} \quad \frac{1}{2} F_{\det A} = \mu(\Psi),$$

for a pair $(\Psi, A) \in \Gamma(\mathrm{Hom}(E, W)) \times \mathscr{A}^s(\mathfrak{w})$, and the Dirac equation for the field ξ. The first equation was studied by Bryan and Wentworth [BW96] and Haydys and Walpuski [HW15]. Building on the work of Haydys and Walpuski, in the next chapter we develop a theory of counting solutions of this equation, in the case $r = 2$, and study how the signed count of solutions changes when we vary the parameters of the equation.

3 FUETER SECTIONS WITH SINGULARITIES

In the previous chapter, we studied the relation between ρ–monopoles and Fueter sections. This led to the discovery of a wall-crossing phenomenon for ρ–monopoles, described by Theorem 2.8.1. In this chapter, we study the wall-crossing phenomenon in detail for the *Seiberg–Witten equation with two spinors*, which is closely related to the $\mathrm{ADHM}_{2,1}$ Seiberg–Witten equation introduced earlier. The main result is a wall-crossing formula, stated in Theorem 3.5.7, which describes the change in the signed count of monopoles with two spinors caused by (non-singular) Fueter sections. As an application, we prove Theorem 3.1.7, which asserts the existence of *singular* Fueter sections with values in $\mathbf{H}/\mathbf{Z}_2$ on every 3–manifold with $b_1 > 1$. Such singular Fueter sections are examples of *harmonic* $\mathbf{Z}_2$ *spinors*, as defined by Taubes [Tau14]. In particular, Theorem 3.1.7 produces the first examples of singular harmonic $\mathbf{Z}_2$ spinors which are not obtained by means of complex geometry.

The wall-crossing phenomenon is related to an observation made by Joyce [Joy17, Section 8.4] which points out potential issues with the Donaldson–Segal program for counting G_2–instantons. We discuss this in Section 3.11.

REFERENCES. This chapter is based almost entirely on the article [DW17c] written in collaboration with Thomas Walpuski. In order to increase readability, we removed some technical details related to compactness and orientations, providing references to the relevant sections of the article. The discussion of transversality and reducibles in Section 3.2 is taken from [Doa17b].

3.1 EXISTENCE OF HARMONIC $\mathbf{Z}_2$ SPINORS

The notion of a harmonic $\mathbf{Z}_2$ spinor was introduced by Taubes [Tau14] as an abstraction of various limiting objects appearing in compactifications of moduli spaces of flat $\mathbf{PSL}_2(\mathbf{C})$–connections over 3–manifolds [Tau13b] and solutions to the Kapustin–Witten equation [Tau13a], the Vafa–Witten equation [Tau17], and the Seiberg–Witten equation with multiple spinors [HW15; Tau16]. All of these equations are examples of Seiberg–Witten equations associated with quaternionic representations. In this chapter, we focus on the last of these examples.

Definition 3.1.1. Let M be a closed Riemannian manifold and $\mathbf{S}$ a Dirac bundle over M.[1] Denote by $\not{D}\colon \Gamma(\mathbf{S}) \to \Gamma(\mathbf{S})$ the associated Dirac operator. A $\mathbf{Z}_2$ *spinor* with values in $\mathbf{S}$ is a triple $(Z, \mathfrak{l}, \Psi)$ consisting of:

1. a proper closed subset $Z \subset M$,
2. a Euclidean line bundle $\mathfrak{l} \to M\backslash Z$, and
3. a section $\Psi \in \Gamma(M\backslash Z, \mathbf{S} \otimes \mathfrak{l})$

1 A Dirac bundle is a bundle of Clifford modules together with a metric and a compatible connection; see, [LM89, Chapter II, Definition 5.2].

such that $|\Psi|$ extends to a Hölder continuous function on M with $|\Psi|^{-1}(0) = Z$ and $|\nabla\Psi| \in L^2(M\backslash Z)$. We say that $(Z, \mathfrak{l}, \Psi)$ is *singular* if $\mathfrak{l}$ does not extend to a Euclidean line bundle on M. A $\mathbf{Z}_2$ spinor $(Z, \mathfrak{l}, \Psi)$ is called *harmonic* if

$$\not{D}\Psi = 0$$

holds on $M\backslash Z$.

Remark 3.1.2. Taubes [Tau14] proved that if the dimension of M is 3 or 4, then Z has Hausdorff codimension at least 2. More recently, Zhang [Zha17] proved that Z is, in fact, rectifiable.

Remark 3.1.3. If $\mathfrak{l}$ extends to a Euclidean line bundle on M, then $\mathbf{S} \otimes \mathfrak{l}$ extends to a Dirac bundle on M and Ψ extends to a harmonic spinor defined on all of M which takes values in $\mathbf{S} \otimes \mathfrak{l}$ and vanishes precisely along Z.

Remark 3.1.4. Away from the subset Z, a harmonic $\mathbf{Z}_2$ spinor can be interpreted as a Fueter section of the bundle $(\mathbf{S}\backslash\{0\})/\mathbf{Z}_2$ whose fiber is the hyperkähler manifold $(\mathbf{H}^r\backslash\{0\})/\mathbf{Z}_2$, with $r = \mathrm{rank}_{\mathbf{H}}\, \mathbf{S}$. By the third condition in Definition 3.1.1, this section extends to a continuous section, defined over all of M, of the bundle $\mathbf{S}/\mathbf{Z}_2$ whose fiber is a singular hyperkähler variety $\mathbf{H}^r/\mathbf{Z}_2$. The resulting section takes values in the singular stratum $\{0\} \subset \mathbf{H}^r/\mathbf{Z}_2$ precisely along Z. Thus, a singular harmonic $\mathbf{Z}_2$ spinor is an example of a singular Fueter section.

We know from Chapter 2 that non-singular (that is: defined over the entire 3–manifold) Fueter sections play an important role in the study of Seiberg–Witten equations associated with quaternionic representations. The goal of this chapter is to argue that singular Fueter sections do exist and that we should expect them to play a similar role.

The harmonic $\mathbf{Z}_2$ spinors appearing as limits of flat $\mathbf{P}\mathrm{SL}_2(\mathbf{C})$–connections over a 3–manifold M take values in the bundle $\underline{\mathbf{R}} \oplus T^*M$ equipped with the Dirac operator

$$\not{D} = \begin{pmatrix} 0 & \mathrm{d}^* \\ \mathrm{d} & *\mathrm{d} \end{pmatrix}.$$

The harmonic $\mathbf{Z}_2$ spinors appearing as limits of the Seiberg–Witten equation with two spinors in dimension three take values in the Dirac bundle

$$\mathbf{S} = \mathrm{Re}(S \otimes E).$$

This bundle is constructed as follows. Denote by S the spinor bundle of a spin structure $\mathfrak{s}$ on the 3–manifold M. Denote by E a rank two Hermitian bundle with trivial determinant line bundle $\Lambda^2_{\mathbf{C}}E$ and equipped with a compatible connection. Both S and E are quaternionic vector bundles and thus have complex anti-linear endomorphism j_S, j_E satisfying $j_S^2 = -\mathrm{id}_S$ and $j_E = -\mathrm{id}_E$; see also Section 3.3. These endow the complex vector bundle $S \otimes E$ with a real structure: $\overline{s \otimes e} := j_s s \otimes j_E e$. The $\mathbf{C}$–linear Dirac operator acting on $\Gamma(S \otimes E)$ preserves $\Gamma(\mathrm{Re}(S \otimes E))$ and gives rise to an $\mathbf{R}$–linear Dirac operator acting on $\Gamma(\mathrm{Re}(S \otimes E))$.

Henceforth, we specialize situation described in the previous paragraph. The Dirac operator on $\mathrm{Re}(S \otimes E)$, and thus also the notion of a harmonic $\mathbf{Z}_2$ spinor, depends on the choice of a Riemannian metric on M and a connection on E.

Definition 3.1.5. Let $\mathscr{M}et(M)$ be the space of Riemannian metrics on M and $\mathscr{A}(E)$ the space of SU(2) connections on E. The *space of parameters* is

$$\mathscr{P} := \mathscr{M}et(M) \times \mathscr{A}(E)$$

equipped with the C^∞ topology. Given a spin structure $\mathfrak{s}$ on M and $\mathbf{p} \in \mathscr{P}$, we denote by $\not{D}^{\mathfrak{s}}_{\mathbf{p}}$ the corresponding Dirac operator on $\Gamma(\mathrm{Re}(S \otimes E))$. We will say that a triple $(Z, \mathfrak{l}, \Psi)$ is a harmonic $\mathbf{Z}_2$ spinor *with respect to* $\mathbf{p}$ if it satisfies the conditions of Definition 3.1.1 with $\not{D} = \not{D}^{\mathfrak{s}}_{\mathbf{p}}$.

Question 3.1.6. For which parameters $\mathbf{p} \in \mathscr{P}$ does there exist a singular harmonic $\mathbf{Z}_2$ spinor with respect to $\mathbf{p}$?

The answer to this question for non-singular harmonic $\mathbf{Z}_2$ spinors (that is: harmonic spinors) is well-understood. Let $\mathscr{W}^{\mathfrak{s}}$ be the set of $\mathbf{p} \in \mathscr{P}$ for which $\dim\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} > 0$. It is the closure of $\mathscr{W}^{\mathfrak{s}}_1$, the set of $\mathbf{p}$ for which $\dim\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} = 1$. Moreover, $\mathscr{W}^{\mathfrak{s}}_1$ is a cooriented, codimension one submanifold of $\mathscr{P}$ and $\mathscr{W}^{\mathfrak{s}} \backslash \mathscr{W}^{\mathfrak{s}}_1$ has codimension three. (See Proposition 3.10.2 and Proposition 3.10.6.) The intersection number of a path $(\mathbf{p}_t)_{t\in[0,1]}$ with $\mathscr{W}^{\mathfrak{s}}_1$ is given by the *spectral flow* of the path of operators $(\not{D}^{\mathfrak{s}}_{\mathbf{p}_t})_{t\in[0,1]}$, defined by Atiyah, Patodi, and Singer [APS76, Section 7]. Therefore, along any path with non-zero spectral flow there exists a parameter $\mathbf{p}_\star$ such that $\dim\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_\star} > 0$. Moreover, if the path is generic,[2] then the kernel is spanned by a nowhere vanishing spinor (because $\dim M < \operatorname{rank} \mathbf{S}$; see Definition 3.5.1 and Proposition 3.9.1).

By contrast, little is known about the existence of singular harmonic $\mathbf{Z}_2$ spinors. The only examples known thus far have been obtained by means of complex geometry, on Riemannian 3–manifolds of the form $M = S^1 \times \Sigma$ for a Riemann surface Σ; see [Tau13b, Theorem 1.2] in the case $\mathbf{S} = \underline{\mathbf{R}} \oplus T^*M$ and Chapter 4 in the case $\mathbf{S} = \mathrm{Re}(S \otimes E)$. We remedy this situation by proving—in a rather indirect way—that 3-manifolds abound with singular harmonic $\mathbf{Z}_2$ spinors.

Theorem 3.1.7. *For every closed, connected, oriented 3–manifold M with $b_1(M) > 1$ there exist a $\mathbf{p}_\star \in \mathscr{P}$ and a singular harmonic $\mathbf{Z}_2$ spinor with respect to $\mathbf{p}_\star$. In fact, there is a closed subset $\mathscr{W}_b \subset \mathscr{P}$ and a non-zero cohomology class $\omega \in H^1(\mathscr{P}\backslash\mathscr{W}_b, \mathbf{Z})$ with the property that if $(\mathbf{p}_t)_{t\in S^1}$ is a generic loop in $\mathscr{P}\backslash\mathscr{W}_b$ and*

$$\omega([\mathbf{p}_t]) \neq 0,$$

then there exists a singular harmonic $\mathbf{Z}_2$ spinor with respect to some $\mathbf{p}_\star$ in $(\mathbf{p}_t)_{t\in S^1}$.

Remark 3.1.8. The definition of $\mathscr{W}_b$ is given in Definition 3.10.1 and the precise meaning of a generic loop is given in Definition 3.5.1 and Proposition 3.2.4.

Remark 3.1.9. Theorem 3.1.7 suggests that on 3–manifolds the appearance of singular harmonic $\mathbf{Z}_2$ spinors is a codimension one phenomenon—as is the appearance of harmonic spinors. This is in consensus with the work of Takahashi [Tak15; Tak17], who proved that the linearized deformation theory of singular harmonic $\mathbf{Z}_2$ spinors with $Z = S^1$ is an index zero Fredholm problem (or index minus one, after scaling is taken into account).

2 *Generic* means from a residual subset of the space of objects in question. A subset of a topological space is residual if it contains a countable intersection of open and dense subsets. Baire's theorem asserts that a residual subset of a complete metric space is dense.

Remark 3.1.10. The assumption $b_1(M) > 1$ has to do with reducible solutions to the Seiberg–Witten equation with two spinors. We expect a variant of the theorem to be true for $b_1(M) \in \{0, 1\}$ as well. In this case, one also has to take into account the wall-crossing caused by reducible solutions which was studied in classical Seiberg–Witten theory by Chen [Che97] and Lim [Lim00].

The proof of Theorem 3.1.7 relies on the wall-crossing formula for $n(\mathbf{p})$, the signed count of solutions to the Seiberg–Witten equation with two spinors. The number $n(\mathbf{p})$ is defined provided $\mathbf{p}$ is generic and there are no singular harmonic $\mathbf{Z}_2$ spinors with respect to $\mathbf{p}$. The wall-crossing formula, whose precise statement is Theorem 3.5.7, can be described as follows. Let $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ be the set of $\mathbf{p} \in \mathscr{P}$ for which $\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} = \mathbf{R}\langle \Psi \rangle$ with Ψ nowhere vanishing, and let $\mathscr{W}_{1,\varnothing}$ be the union of all $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ for all spin structures $\mathfrak{s}$. There is a closed subset $\mathscr{W}_b \subset \mathscr{P}$, as in Theorem 3.1.7, such that $\mathscr{W}_{1,\varnothing}$ is a closed, cooriented, codimension one submanifold of $\mathscr{P} \backslash \mathscr{W}_b$. If $(\mathbf{p}_t)_{t \in [0,1]}$ is a generic path in $\mathscr{P} \backslash \mathscr{W}_b$ and there are no singular harmonic $\mathbf{Z}_2$ spinors with respect to any $\mathbf{p}_t$ with $t \in [0, 1]$, then the difference

$$n(\mathbf{p}_1) - n(\mathbf{p}_0)$$

is equal to the intersection number of the path $(\mathbf{p}_t)_{t \in [0,1]}$ with $\mathscr{W}_{1,\varnothing}$. In particular, if $(\mathbf{p}_t)_{t \in S^1}$ is a generic loop whose intersection number with $\mathscr{W}_{1,\varnothing}$ is non-zero, then there must be a singular harmonic $\mathbf{Z}_2$ spinor for some $\mathbf{p}_\star$ in $(\mathbf{p}_t)$.

Remark 3.1.11. Although the wall-crossing for $n(\mathbf{p})$ does occur when the spectrum of $\not{D}^{\mathfrak{s}}_{\mathbf{p}}$ crosses zero, the contribution of a nowhere vanishing harmonic spinor to the wall-crossing formula is not given by the sign of the spectral crossing but instead by the mod 2 topological degree of the harmonic spinor, as in Definition 3.5.6. Therefore, the wall-crossing formula is not given by the spectral flow. This should be contrasted with the wall-crossing phenomenon for the classical Seiberg–Witten equation caused by reducible solutions, as in Remark 3.1.10. Indeed, the wall-crossing described in this paper is a result of the non-compactness of the moduli spaces of solutions and as such it is a new phenomenon, with no counterpart in classical Seiberg–Witten theory.

The cohomology class $\omega \in H^1(\mathscr{P} \backslash \mathscr{W}_b, \mathbf{Z})$ in Theorem 3.1.7 is defined by intersecting loops in $\mathscr{P} \backslash \mathscr{W}_b$ with $\mathscr{W}_{1,\varnothing}$. We prove that ω is non-trivial, by exhibiting a loop $(\mathbf{p}_t)_{t \in S^1}$ on which ω evaluates as ± 2. In particular, $\mathscr{P} \backslash \mathscr{W}_b$ is not simply-connected and we can take $(\mathbf{p}_t)_{t \in S^1}$ to be a small loop linking $\mathscr{W}_b$.

3.1.1 *Notation*

Here is a summary of various notations used throughout this chapter:

- We use $\mathfrak{w}$ to denote a spin^c structure. The associated complex spinor bundle is denoted by W. The *determinant line bundle* of W is $\det W = \Lambda^2_{\mathbf{C}} W$.
- For every $\mathbf{p} = (g, B)$ in the parameter space $\mathscr{P}$ and for every connection $A \in \mathscr{A}(\det W)$ we write

  $$\not{D}_{A,\mathbf{p}} \colon \Gamma(\mathrm{Hom}(E, W)) \to \Gamma(\mathrm{Hom}(E, W))$$

  for the $\mathbf{C}$–linear Dirac operator induced the connection B on E as well as the spin^c connection on W, determined by A and the Levi–Civita connection of

g. We will suppress the subscript $\mathbf{p}$ from the notation when its presence is not relevant to the current discussion.

- We use $\mathfrak{s}$ (possibly with a subscript: $\mathfrak{s}_0$, $\mathfrak{s}_1$, etc.) to denote a spin structure on M. The associated spinor bundle is denoted by $S_{\mathfrak{s}}$.
- For every $\mathbf{p} = (g, B) \in \mathscr{P}$ we write

$$\not{D}^{\mathfrak{s}}_{\mathbf{p}} \colon \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E)) \to \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E))$$

for the $\mathbf{R}$–linear Dirac operator induced by the spin connection on $S_{\mathfrak{s}}$, associated with the Levi–Civita connection of g, and the connection B on E.

3.2 THE SEIBERG–WITTEN EQUATION WITH TWO SPINORS

Let M be a compact, oriented 3–manifold and let $E \to M$ be a Hermitian vector bundle of rank two, with a fixed trivialization of the determinant line bundle $\Lambda^2_{\mathbf{C}} E = \mathbf{C}$. Fix a spin^c structure $\mathfrak{w}$ and denote its spinor bundle by W. Set

$$\mathscr{Z} := \ker\left(\mathrm{d} \colon\ \Omega^2(M, i\mathbf{R}) \to \Omega^3(M, i\mathbf{R})\right). \tag{3.2.1}$$

Definition 3.2.1. Let $\mathbf{p} = (g, B) \in \mathscr{P}$ and $\eta \in \mathscr{Z}$. The *η–perturbed Seiberg–Witten equation with two spinors* is the following differential equation for $(\Psi, A) \in \Gamma(\mathrm{Hom}(E, W)) \times \mathscr{A}(\det W)$:

$$\begin{aligned} &\not{D}_A \Psi = 0 \quad \text{and} \\ &\frac{1}{2} F_A + \eta = \mu(\Psi). \end{aligned} \tag{3.2.2}$$

Here $\not{D}_A = \not{D}_{A,\mathbf{p}}$ is the $\mathbf{C}$–linear Dirac operator on $\mathrm{Hom}(E, W)$ and

$$\mu(\Psi) = \mu_{\mathbf{p}}(\Psi) := \Psi\Psi^* - \frac{1}{2}|\Psi|^2 \,\mathrm{id}_W$$

is a section of $i\mathfrak{su}(W)$ which is identified with an element of $\Omega^2(M, i\mathbf{R})$ using the Clifford multiplication. Both equations depend on the choice of $\mathbf{p}$ and the second equation depends also on the choice of η.[3]

Let $\mathscr{G}(\det W)$ be the gauge group of $\det W$. For $(\mathbf{p}, \eta) \in \mathscr{P} \times \mathscr{Z}$, we denote by

$$\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta) := \left\{ [\Psi, A] \in \frac{\Gamma(\mathrm{Hom}(E, W)) \times \mathscr{A}(\det W)}{\mathscr{G}(\det W)} : \begin{array}{l} (\Psi, A) \text{ satisfies (3.2.2)} \\ \text{with respect to } \mathbf{p} \text{ and } \eta \end{array} \right\}$$

the *moduli space* of solutions to (3.2.2).

As discussed in Section 2.3, the infinitesimal deformation theory of (3.2.2) around a solution (Ψ, A), is controlled by the linear operator

$$\begin{aligned} L_{\Psi,A} = L_{\Psi,A,\mathbf{p}} \colon\ &\Gamma(\mathrm{Hom}(E, W)) \oplus \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}) \\ &\to \Gamma(\mathrm{Hom}(E, W)) \oplus \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}) \end{aligned}$$

3 While the equation (3.2.2) makes sense for any $\eta \in \Omega^2(M, i\mathbf{R})$, it is an easy exercise to show that the existence of a solution implies that $\mathrm{d}\eta = 0$. Thus, we consider only $\eta \in \mathscr{Z}$.

defined by

$$L_{\Psi,A,\mathbf{p}} := \begin{pmatrix} -\not{D}_{A,\mathbf{p}} & -\mathfrak{a}_{\Psi,\mathbf{p}} \\ -\mathfrak{a}^*_{\Psi,\mathbf{p}} & \mathfrak{d}_{\mathbf{p}} \end{pmatrix} \tag{3.2.3}$$

with

$$\mathfrak{d} = \mathfrak{d}_{\mathbf{p}} := \begin{pmatrix} *\mathrm{d} & \mathrm{d} \\ \mathrm{d}^* & \end{pmatrix} \quad \text{and} \quad \mathfrak{a}_\Psi = \mathfrak{a}_{\Psi,\mathbf{p}} := \begin{pmatrix} \bar{\gamma}(\cdot)\Psi & \rho(\cdot)\Psi \end{pmatrix}.^4 \tag{3.2.4}$$

Here $\bar{\gamma}$ is the Clifford multiplication by elements of $T^*M \otimes i\mathbf{R}$ and ρ is the linearized action of the gauge group: pointwise multiplication by elements of $i\mathbf{R}$. The Hodge star operator $*$ and $\bar{\gamma}$ both depend on $\mathbf{p}$, but we have suppressed this dependence in the notation.

Definition 3.2.2. We say that a solution (Ψ, A) of (3.2.2) is *irreducible* if $\Psi \neq 0$, and *unobstructed* if $L_{\Psi,A,\mathbf{p}}$ is invertible.

If (Ψ, A) is irreducible and unobstructed, then it represents an isolated point in $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$; see Section 2.3.

Remark 3.2.3. If all solutions are irreducible, but possibly obstructed, the existence of local Kuranishi models constructed in Proposition 2.3.1 allows us to equip $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ with the structure of a real analytic space. Indeed, since the Seiberg–Witten equation involves only linear and quadratic operations, we can choose the Kuranishi map $\mathrm{ob} = \mathrm{ob}(\mathbf{p}, \eta, \cdot)$ to be analytic. Thus, $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is locally homeomorphic to the real analytic set $V = \mathrm{ob}^{-1}(0)$. As we vary ob and V, the rings of analytic functions $\mathcal{O}/I(V)$ glue to a globally defined structure sheaf making $\mathfrak{M}_{\mathbf{w}}(\mathbf{p}, \eta)$ into a real analytic space; see [FM94, Sections 4.1.3-4.1.4] for a detailed construction of the real analytic structure on the moduli space.

Proposition 3.2.4. *If $b_1(M) > 0$, then for every $(\mathbf{p}, \eta)$ from a residual subset of $\mathscr{P} \times \mathscr{Z}$ all solutions to (3.2.2) are irreducible and unobstructed.*

Remark 3.2.5. The same statement is true for the Seiberg–Witten equation with n spinors obtained by replacing E by an $\mathrm{SU}(n)$–bundle. The proof is the same as in the case $n = 2$.

The proof of Proposition 3.2.4 requires discussing transversality and reducibles.

3.2.1 *Transversality*

The first step in proving Proposition 3.2.4 is to show the following statement using the Sard–Smale theorem.

Proposition 3.2.6. *For every $(\mathbf{p}, \eta)$ from a residual subset of $\mathscr{P} \times \mathscr{Z}$ all irreducible solutions to (3.2.2) are unobstructed.*

Proof. Consider the universal Seiberg–Witten map

$$\hat{S}\colon \mathscr{P} \times \mathscr{Z} \times \Gamma(\mathrm{Hom}(E, W)) \times \mathscr{A}(\det W) \to \Gamma(\mathrm{Hom}(E, W)) \times \Omega^1(M, i\mathbf{R})$$
$$(\mathbf{p}, \eta, \Psi, A) \mapsto (\not{D}_{A,\mathbf{p}}\Psi, *\left(\frac{1}{2}F_A + \eta - \mu(\Psi)\right)).$$

The Seiberg–Witten equation is equivariant with respect to the action of the gauge group $\mathscr{G}(\det W)$, and so $\hat{S}$ descends to a smooth map between the corresponding quotient spaces. Denote this map by S. The zero set of the resulting map is the moduli space $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$.

In what follows we replace $\mathscr{P}$ and $\mathscr{Z}$ by the spaces of C^k parameters and differential forms. We also replace the spaces of smooth connections and sections with their Sobolev completions. Thus, $\mathfrak{S}$ can be considered as a smooth Fredholm map between Banach manifold. If we can show that its derivative is surjective at all irreducible solutions $[\Psi, A]$, then Proposition 3.2.6 will follow, with $\mathscr{P}$ and $\mathscr{Z}$ equipped with the C^k topology. The corresponding statement for the C^∞ topology follows then using standard arguments, cf. [MS12, Theorem 3.1.5].

The partial derivative of S at $(\mathbf{p}, \eta, [\Psi, A])$ obtained by varying $[\Psi, A]$ with $\mathbf{p}$ and $\mathscr{Z}$ fixed can be identified with the operator $L_{\Psi,A,\mathbf{p}}$ given by (3.2.3). Note that this operator acts on the space

$$\Gamma(\mathrm{Hom}(E, W) \oplus \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}),$$

where the last factor appears because we have divided both the domain and codomain of $\hat{S}$ by the action of the gauge group (in other words, we have introduced gauge fixing and gauge co-fixing conditions). We will prove that $\mathrm{d}S_{(\mathbf{p},\eta,[\Psi,A])}$ is surjective by showing that an element $(\hat{\psi}, \hat{a}, \hat{\xi})$ of the above space which is L^2–orthogonal to the image of $\mathrm{d}S_{(\mathbf{p},\eta,[\Psi,A])}$ is necessarily zero. Such an element is, in particular, in the kernel of $L_{\Psi,A,\mathbf{p}}$, which is equivalent to

$$\begin{aligned} \not{D}_{A,\mathbf{p}}\hat{\psi} + \bar{\gamma}(\hat{a})\Psi + \rho(\hat{\xi})\Psi &= 0, \\ *\mu(\Psi, \hat{\psi}) + *\mathrm{d}\hat{a} + \mathrm{d}\hat{\xi} &= 0, \\ \rho^*(\Psi^*\hat{\psi}) + \mathrm{d}^*\hat{a} &= 0. \end{aligned} \tag{3.2.5}$$

Applying d^* to the second equation above and using formula (2.3.2) from Proposition 2.3.9, we arrive at

$$\mathrm{d}^*\mathrm{d}\hat{\xi} + \rho^*(\Psi^*\rho(\hat{\xi})\Psi) = 0.$$

which, after taking the L^2 inner product with $\hat{\xi}$, implies $\mathrm{d}\hat{\xi} = 0$ and $\rho(\hat{\xi})\Psi = 0$. Since $\Psi \neq 0$ by the assumption that $[\Psi, A]$ is irreducible, we have $\hat{\xi} = 0$. Similarly, applying d to the second and third equation above and using formula (2.3.3) from Proposition 2.3.9, and taking the L^2 inner product with $\hat{a}$, we arrive at

$$\mathrm{d}\hat{a} = 0, \quad \mathrm{d}^*\hat{a} = 0, \quad \text{and} \quad \bar{\gamma}(\hat{a})\Psi = 0,$$

which implies $\hat{a} = 0$. Thus, (3.2.5) simplifies to

$$\begin{aligned} \not{D}_{A,\mathbf{p}}\hat{\psi} &= 0, \\ *\mu(\Psi, \hat{\psi}) &= 0, \\ \rho^*(\Psi^*\hat{\psi}) &= 0. \end{aligned} \tag{3.2.6}$$

The partial derivative of S at $(\mathbf{p}, \eta, [\Psi, A])$ obtained by varying B in the parameter $\mathbf{p} = (g, B) \in \mathscr{P}$ is given by

$$b \mapsto \bar{\gamma}(b)\Psi,$$

where $b \in \Omega^1(M, \mathfrak{su}(E))$ is a variation of B, and $\bar{\gamma}$ denotes the Clifford multiplication combined with the action of $\mathfrak{su}(E)$ on E. Thus, the condition that $(\hat{\psi}, 0, 0)$ is L^2–orthogonal to the image of $\mathrm{d}S_{(\mathbf{p},\eta,[\Psi,A])}$ implies

$$\langle \bar{\gamma}(b)\Psi, \hat{\psi}\rangle_{L^2} = 0 \tag{3.2.7}$$

for all $b \in \Omega^1(M, \mathfrak{su}(E))$. We will prove that this implies $\hat{\psi} = 0$. Suppose by contradiction that $\hat{\psi}$ is not identically zero. By the unique continuation theorem for harmonic spinors [Bär97, Corollary 3], the set $\{|\hat{\psi}| > 0\}$ is open and dense in M. The same is true for Ψ and so there exists $x \in M$ such that $|\Psi(x)| > 0$ and $|\hat{\psi}(x)| > 0$. Hence, we can find $b(x) \in T^*_x M \otimes \mathfrak{su}(E_x)$ such that

$$\langle b(x)\Psi(x), \hat{\psi}(x)\rangle > 0.$$

This is an elementary fact of linear algebra; see Proposition 3.2.7 below. Using a bump function, we extend $b(x)$ to $b \in \Omega^1(M, \mathfrak{su}(E))$ violating (3.2.7). The contradiction shows that $\hat{\psi} = 0$ and so $\mathrm{d}S_{(\mathbf{p},\eta,[\Psi,A])}$ is surjective. The statement of the proposition follows then from the Sard–Smale theorem. ☐

Proposition 3.2.7. *Let $n \geqslant 2$ and $V_n = \mathbf{C}^n$ be the standard representation of $\mathrm{SU}(n)$. For every pair of non-zero $v, w \in V_2 \otimes V_n$ there is $b \in \mathfrak{su}(2) \otimes \mathfrak{su}(n)$ such that*

$$\langle bv, w\rangle > 0.$$

Proof. The proof is similar to that of [Ang96, Theorem 1.5]. Let $e_1, \dots, e_n$ be an orthonormal basis of V_n. Write v and w as

$$v = \sum_{i=1}^n v_i \otimes e_i \quad \text{and} \quad w = \sum_{i=1}^n w_i \otimes e_i,$$

for $v_i, w_i \in V_2$. Likewise, denoting by $\sigma_1, \sigma_2, \sigma_3$ the standard basis of $\mathfrak{su}(2)$, we can write b as

$$b = \sum_{k=1}^3 \sigma_k \otimes b_k$$

for some $b_k \in \mathfrak{su}(n)$, so that

$$\langle bv, w\rangle = \sum_{k=1}^3 \sum_{i,j} \langle \sigma_k v_i, w_j\rangle \langle e_i, b_k e_j\rangle.)$$

Suppose by contradiction that $(bv, w) = 0$ for all $b \in \mathfrak{su}(2) \otimes \mathfrak{su}(n)$. In particular, setting b_k to be elementary off-diagonal anti-Hermitian matrices, we see that for $k = 1, 2, 3$ and $i \neq j$

$$\langle \sigma_k v_i, w_j\rangle - \langle \sigma_k v_j, w_i\rangle = 0,$$
$$\langle \sigma_k v_i, w_j\rangle + \langle \sigma_k v_j, w_i\rangle = 0.$$

Hence,

$$\langle \sigma_k v_i, w_j\rangle = 0$$

for $k = 1, 2, 3$ and $i \neq j$. Suppose without loss of generality that $v_1 \neq 0$. Then $\sigma_1 v_1, \sigma_2 v_1, \sigma_3 v_1$ are linearly independent over $\mathbf{R}$ and thus span V_2 over

C. It follows that $w_j = 0$ for $j = 2, 3, \ldots n$. On the other hand, setting $b_k = \mathrm{diag}(1, -1, 0, \ldots, 0) \in \mathfrak{su}(n)$, we obtain that for $k = 1, 2, 3$

$$\langle \sigma_k v_1, w_1 \rangle = 0,$$

which shows that $w_1 = 0$ and so $w = 0$, yielding a contradiction. □

Remark 3.2.8. The proof of Proposition 3.2.6 shows that if $L_{\Psi,A,\mathbf{p}}$ fails to be invertible at a solution $[\Psi, A]$, then its kernel consists of solutions $\hat{\psi}$ to (3.2.6).

3.2.2 *Reducible solutions*

The moduli space $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ might contain reducible solutions at which it develops singularities. In this paper we focus on the favourable case when reducibles can be avoided. As in the classical setting [Limoo, Lemma 14], this is guaranteed by the condition $b_1(M) > 0$.

Proposition 3.2.9. *The subset of those $(\mathbf{p}, \eta) \in \mathscr{P} \times \mathscr{Z}$ for which $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ contains a reducible solution is contained in a closed affine subspace of codimension $b_1(M)$.*

Proof. If (Ψ, A) is a reducible solution of (3.2.2), then $F_A = 2\eta$ and passing to the de Rham cohomology we obtain

$$[\eta] = -\pi i c_1(\mathfrak{w}) \in H^2(M, i\mathbf{R}).$$

Consider the affine subspace of $\mathscr{Z}$ given by

$$V = \{\eta \in \mathscr{Z} \mid [\eta] = -\pi i c_1(\mathfrak{w})\}.$$

In other words, V is the preimage of $-\pi i c_1(\mathfrak{w})$ under the projection $\pi\colon \mathscr{Z} \to H^2(M, i\mathbf{R})$ associating to each closed form its de Rham class. The map π is linear and surjective. It is also continuous because by the Hodge decomposition theorem it is continuous with respect to the $W^{k,2}$–topology for all k. Therefore, V is a closed affine subspace of codimension $\dim H^2(M, i\mathbf{R}) = \dim H^1(M, i\mathbf{R}) = b_1(M)$. □

Proposition 3.2.4 follows immediately from Proposition 3.2.6 combined with Proposition 3.2.9.

3.2.3 *Orientations*

For every $(\mathbf{p}, \eta)$ as in Proposition 3.2.4, the moduli space $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is a zero-dimensional manifold, that is: a collection of isolated points. It follows from the general discussion of Section 2.4 that there is a consistent way of orienting each of the points in the moduli space. Since orientations play an important role in what follows, below we discuss in detail the orientation procedure for the Seiberg–Witten equation with two spinors.

Proposition 3.2.10. *Let $(\Psi_t, A_t, \mathbf{p}_t)_{t\in[0,1]}$ be a path in $\Gamma(\mathrm{Hom}(E, W)) \times \mathscr{A}(\det W) \times \mathscr{P}$ The value of the spectral flow*

$$\mathrm{SF}\left((L_{\Psi_t, A_t, \mathbf{p}_t})_{t\in[0,1]} \right) \in \mathbf{Z}$$

only depends on $(\Psi_0, A_0, \mathbf{p}_0)$ and $(\Psi_1, A_1, \mathbf{p}_1)$.

Remark 3.2.11. If $(D_t)_{t\in[0,1]}$ is a path of self-adjoint Fredholm operators with D_0 or D_1 not invertible, we define $\mathrm{SF}\left((D_t)_{t\in[0,1]}\right) := (D_t+\lambda)_{t\in[0,1]}$ for $0<\lambda\ll 1$. This convention was introduced by Atiyah, Patodi, and Singer [APS76, Section 7].

Proof of Proposition 3.2.10. Since the spectral flow is homotopy invariant, this is a consequence of the fact that $\Gamma(\mathrm{Hom}(E,W))\times\mathscr{A}(\det W)\times\mathscr{P}$ is contractible. □

Definition 3.2.12. For $(\Psi_0,A_0,\mathbf{p}_0)$ and $(\Psi_1,A_1,\mathbf{p}_1)\in\Gamma(\mathrm{Hom}(E,W))\times\mathscr{A}(\det W)\times\mathscr{P}$, we define the *orientation transport*

$$\mathrm{OT}\left((\Psi_0,A_0,\mathbf{p}_0),(\Psi_1,A_1,\mathbf{p}_1)\right) := (-1)^{\mathrm{SF}\left((L_{\Psi_t,A_t,\mathbf{p}_t})_{t\in[0,1]}\right)},$$

for path $(\Psi_t,A_t,\mathbf{p}_t)_{t\in[0,1]}$ from $(\Psi_0,A_0,\mathbf{p}_0)$ to $(\Psi_1,A_1,\mathbf{p}_1)$.

Remark 3.2.13. The orientation transport can be alternatively defined using the determinant line bundle of the family of Fredholm operators $(L_{\Psi,A,\mathbf{p}})$ as $(\Psi,A,\mathbf{p})$ varies in $\Gamma(\mathrm{Hom}(E,W))\times\mathscr{A}(\det W)\times\mathscr{P}$. This point of view is explained in detail in [DW17c, Appendix B].

Since the spectral flow is additive with respect to path composition, we have

$$\begin{aligned}&\mathrm{OT}\left((\Psi_0,A_0,\mathbf{p}_0),(\Psi_2,A_2,\mathbf{p}_2)\right)\\&\quad=\mathrm{OT}\left((\Psi_0,A_0,\mathbf{p}_0),(\Psi_1,A_1,\mathbf{p}_1)\right)\cdot\mathrm{OT}\left((\Psi_1,A_1,\mathbf{p}_1),(\Psi_2,A_2,\mathbf{p}_2)\right).\end{aligned}\tag{3.2.8}$$

Proposition 3.2.14.

1. *For every $(A_0,\mathbf{p}_0)$ and $(A_1,\mathbf{p}_1)\in\mathscr{A}(\det W)\times\mathscr{P}$, we have*

$$\mathrm{OT}((0,A_0,\mathbf{p}_0),(0,A_1,\mathbf{p}_1)) = +1.$$

2. *For every $(\Psi,A,\mathbf{p})\in\Gamma(\mathrm{Hom}(E,W))\times\mathscr{A}(\det W)\times\mathscr{P}$ and every $u\in\mathscr{G}(\det W)$, we have*

$$\mathrm{OT}((\Psi,A,\mathbf{p}),(u\cdot\Psi,u\cdot A,\mathbf{p})) = +1.$$

Proof. Observe that

$$L_{0,A_i,\mathbf{p}_i} = \not{D}_{A_i,\mathbf{p}_i}\oplus\mathfrak{d}_{\mathbf{p}_i}$$

where $\not{D}_{A_i,\mathbf{p}_i}$ is a complex-linear Dirac operator on $\mathrm{Hom}(E,W)$ and $\mathfrak{d}_{\mathbf{p}_i}$ is defined in (3.2.4). Let $(A_t,\mathbf{p}_t)_{t\in[0,1]}$ be a path in $\mathscr{A}(\det W)\times\mathscr{P}$ joining $(A_0,\mathbf{p}_0)$ and $(A_1,\mathbf{p}_1)$. The spectral flow of $(\not{D}_{A_t,\mathbf{p}_t})_{t\in[0,1]}$ is even because the operators $\not{D}_{A_t,\mathbf{p}_t}$ are complex linear. The spectral flow of $(\mathfrak{d}_{\mathbf{p}_t})_{t\in[0,1]}$ is trivial because the dimension of the kernel of $\mathfrak{d}_{\mathbf{p}_t}$ is $1+b_1(M)$ and does not depend on $t\in[0,1]$. This proves item 1. To prove item 2, choose a path $(\Psi_t,A_t)_{t\in[0,1]}$ from (Ψ,A) to $(u\cdot\Psi,u\cdot A)$. The spetral flow $\mathrm{SF}(L_{\Psi_t,A_t})$ can be computed to be even using a theorem of Atiyah–Patodi–Singer [APS76, Section 7], in the same way as in the proof of Proposition 2.4.1. □

Proposition 3.2.14 and (3.2.8) show that the following definition is independent of any of the choices being made.

Definition 3.2.15. For $[\Psi,A]\in\mathfrak{M}_{\mathfrak{w}}(\mathbf{p},\eta)$, we define

$$\mathrm{sign}[\Psi,A] := \mathrm{OT}((0,A_0,\mathbf{p}_0),(\Psi,A,\mathbf{p})),$$

for any choice of $(A_0,\mathbf{p}_0)\in\mathscr{A}(\det W)\times\mathscr{P}$.

3.2.4 *Counting solutions: unobstructed case*

As we will see, if $(\mathbf{p}, \eta)$ is generic, and under the assumption that there are no harmonic $\mathbf{Z}_2$ spinors with respect to $\mathbf{p}$, then $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is a compact, oriented, zero-dimensional manifold, that is: a finite set of points with prescribed signs.

Definition 3.2.16. Suppose that $(\mathbf{p}, \eta) \in \mathscr{P} \times \mathscr{Z}$ is chosen so that all solutions to (3.2.2) are irreducible and unobstructed, and there are no harmonic $\mathbf{Z}_2$ spinors with respect to $\mathbf{p}$. In this situation, we define

$$n_{\mathfrak{w}}(\mathbf{p}, \eta) := \sum_{[\Psi, A] \in \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)} \operatorname{sign}[\Psi, A], \tag{3.2.9}$$

the signed count of solutions to the Seiberg–Witten equation with two spinors.

3.2.5 *Counting solutions: Zariski smooth case*

Sometimes we face a situation in which the moduli space $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ consists of obstructed solutions but nevertheless has the structure of a smooth manifold.

Definition 3.2.17. The moduli space $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is said to be *Zariski smooth* if it consists of irreducible solutions and for every $[\Psi, A] \in \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ there the map $\operatorname{ob}(\mathbf{p}, \eta, \cdot)$ constructed in Proposition 2.3.1 is zero.

Proposition 3.2.18. *If $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is Zariski smooth, then it is a disjoint uniont of smooth manifolds (of possibly different dimensions). The space $\ker L_{\Psi,A,\mathbf{p}}$ is the tangent space to $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ at $[\Psi, A]$. Similarly, the spaces $\operatorname{coker} L_{\Psi,A,\mathbf{p}}$ form a vector bundle over $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$, as the point $[\Psi, A]$ varies. We call this vector bundle the* obstruction bundle *and denote by $\mathfrak{O}$. The obstruction bundle is isomorphic, as an unoriented real vector bundle, to the cotangent bundle $T\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$.*

The proof is standard and we omit it. Thanks to this proposition, we can extend the definition of $n_{\mathfrak{w}}(\mathbf{p}, \eta)$ to the case when $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is Zariski smooth.

Definition 3.2.19. Suppose that $(\mathbf{p}, \eta) \in \mathscr{P} \times \mathscr{Z}$ is chosen so that the moduli space $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is Zariski smooth and there are no harmonic $\mathbf{Z}_2$ spinors with respect to $\mathbf{p}$. Let $\mathscr{U} \subset \mathscr{P} \times \mathscr{Z}$ be a neighborhood of $(\mathbf{p}, \eta)$ with the property that for all $(\mathbf{p}', \eta') \in \mathscr{U}$, the moduli space $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}', \eta')$ is compact and consists of irreducible solutions; it follows from Proposition 3.2.9 and Theorem 3.4.1 below that such $\mathscr{U}$ exists. Choose $(\mathbf{p}', \eta') \in \mathscr{U}$ so that all solutions in $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}', \eta')$ are in addition unobstructed. We then define

$$n_{\mathfrak{w}}(\mathbf{p}, \eta) := n_{\mathfrak{w}}(\mathbf{p}', \eta')$$

where the right-hand side is as in Definition 3.2.16.

The next proposition shows that the above definition does not depend on the choice of $(\mathbf{p}', \eta')$, and gives two equivalent definition of $n_{\mathfrak{w}}(\mathbf{p}, \eta)$ in the Zariski smooth case. Observe that when $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is Zariski smooth, the orientation procedure described earlier equips the obstruction bundle $\mathfrak{O} \to \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ with a relative orientation, that is: a preferred trivialization of the real line bundle $\det(T\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)) \otimes \det(\mathfrak{O})^*$.

Proposition 3.2.20. *Let $(\mathbf{p}, \eta)$ and $(\mathbf{p}', \eta')$ be as in Definition 3.2.19.*

1. *If s is any transverse section of $\mathfrak{O} \to \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$, then*

$$n(\mathbf{p}', \eta') = \sum_{x \in s^{-1}(0)} \operatorname{sign}(x),$$

where $\operatorname{sign}(x)$ *is obtained using the relative orientation on $\mathfrak{O} \to \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$.*

2. *Suppose that all components of $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ are orientable and orient them arbitrarily. The relative orientation on $\mathfrak{O} \to \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ makes $\mathfrak{O}$ into an oriented real vector bundle. If $e(\mathfrak{O})$ is the Euler class of $\mathfrak{O}$, then*

$$n(\mathbf{p}', \eta') = \int_{\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)} e(\mathfrak{O}).$$

The proof is an easy application of the Poincaré–Hopf index theorem, the Gauss–Bonnet theorem, and [FM94, Lemma 3.3].

3.3 COMPUTATION OF THE HYPERKÄHLER QUOTIENT

The Seiberg–Witten equation with two spinors is the Seiberg–Witten equation associated with the standard representation of $\mathrm{U}(1)$ on the quaternionic vector space $\mathrm{Hom}(\mathbf{C}^2, \mathbf{H})$ In this section we describe $\mathbf{H}/\mathbf{Z}_2$ as the hyperkähler quotient associated with this representation. This construction provides a connection between harmonic $\mathbf{Z}_2$ spinors and Seiberg–Witten monopoles with two spinors, and we will use it in the discussion of compactness and the Haydys correspondence in Section 3.4.

Set

$$S := \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^2, \mathbf{H})$$

with $\mathbf{H}$ considered as a complex vector space whose complex structure is given by right-multiplication with i. S is a quaternionic Hermitian vector space: its $\mathbf{H}$–module structure arises by left-multiplication. The action of $\mathrm{U}(1)$ on S given by $\rho(e^{i\theta})\Psi = e^{i\theta}\Psi$ is a quaternionic representation with associated moment map

$$\mu(\Psi) = \Psi\Psi^* - \frac{1}{2}|\Psi|^2 \operatorname{id}_{\mathbf{H}}.$$

The standard complex volume form $\Omega = e^1 \wedge e^2 \in \Lambda^2(\mathbf{C}^2)^*$ and the standard Hermitian metric on $\mathbf{C}^2$, define a complex anti-linear map $J \colon \mathbf{C}^2 \to \mathbf{C}^2$ by

$$-\langle v, Jw \rangle = \Omega(v, w).$$

This makes $\mathbf{C}^2$ into a $\mathbf{H}$–module.

Proposition 3.3.1. *We have*

$$S^{\mathrm{reg}} /\!\!/\!\!/ \mathrm{U}(1) = \Big(\mathrm{Re}(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2) \backslash \{0\}\Big) / \mathbf{Z}_2.$$

Here the real structure on $\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2$ is given by $\overline{q \otimes v} := qj \otimes Jv$.

Proof. The complex volume form Ω defines a complex linear isomorphism $(\mathbf{C}^2)^* \cong \mathbf{C}^2$; hence, we can identify $S \cong \mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2$. We will further identify

$\mathbf{C}^2$ with $\mathbf{H}$ via $(z,w) \mapsto z + wj$. With respect to this identification the complex structure is given by left-multiplication with i and J becomes left-multiplication by j. If we denote by H_+ (H_-) the quaternions equipped with their right (left) $\mathbf{H}$–module structure, then we can identify S with

$$\mathbf{H}_+ \otimes_{\mathbf{C}} \mathbf{H}_-.$$

In this identification the action of $\mathrm{U}(1)$ is given by

$$\rho(e^{i\theta})q_+ \otimes q_- = q_+ e^{i\theta} \otimes q_- = q_+ \otimes e^{i\theta} q_-,$$

and the moment map becomes

$$\mu(q_1 \otimes 1 + q_2 \otimes j) = -i \otimes \frac{1}{2}(q_1 i \bar{q}_1 + q_2 i \bar{q}_2) \in i\mathbf{R} \otimes \operatorname{Im} \mathbf{H}.$$

If $\mu(q_1 \otimes 1 + q_2 \otimes j) = 0$, then

$$q_1 i \bar{q}_1 = -q_2 i \bar{q}_2. \tag{3.3.1}$$

This implies that $|q_1| = |q_2|$. Unless q_1 and q_2 both vanish, there is a unique $p \in \mathbf{H}$ satisfying

$$|p| = 1 \quad \text{and} \quad q_1 = q_2 p.$$

From (3.3.1), it follows that

$$pi = -ip;$$

hence, $p = je^{i\phi}$ for some $\phi \in \mathbf{R}$. It follows that, for any $\theta \in \mathbf{R}$,

$$q_1 e^{i\theta} = q_2 e^{i\theta} \cdot j e^{i(\phi + 2\theta)}.$$

Since

$$\overline{q_1 \otimes 1 + q_2 \otimes j} = -q_2 j \otimes 1 + q_1 j \otimes j,$$

the real part of $\mathbf{H}_+ \otimes_{\mathbf{C}} \mathbf{H}_-$ consists of those $q_1 \otimes 1 + q_2 \otimes j$ with

$$q_1 = -q_2 j.$$

Consequently, the $\mathrm{U}(1)$–orbit of each non-zero $\mathbf{q} = q_1 \otimes 1 + q_2 \otimes j$ intersects $\mathrm{Re}(\mathbf{H}_+ \otimes_{\mathbf{C}} \mathbf{H}_-)$ twice: in $\pm\rho(e^{i(-\phi/2 + \pi/2)})\mathbf{q}$. □

3.4 COMPACTNESS OF THE MODULI SPACE

In general, $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ might be non-compact; and even if it is compact for given $(\mathbf{p}, \eta)$, compactness might still fail as $(\mathbf{p}, \eta)$ varies. As explained in Section 2.5, this can only happen when, along a sequence of solutions to (3.2.2), the L^2–norm of the spinors goes to infinity. The following result, which is a version of Conjecture 2.5.2 for the Seiberg–Witten equation wth two spinors, describes in which sense one can still take a rescaled limit in such a situation.

Theorem 3.4.1 ([HW15, Theorem 1.5])**.** *Let $(\mathbf{p}_i, \eta_i)$ be a sequence in $\mathscr{P} \times \mathscr{Z}$ which converges to $(\mathbf{p}, \eta)$ in C^∞. Let (Ψ_i, A_i) be a sequence of solutions of (3.2.2) with respect to $(\mathbf{p}_i, \eta_i)$. If $\limsup_{i\to\infty} \|\Psi_i\|_{L^2} = \infty$, then after rescaling $\tilde{\Psi}_i = \Psi_i / \|\Psi_i\|_{L^2}$ and passing to a subsequence the following hold:*

1. *The subset*

$$Z := \left\{ x \in M : \limsup_{i \to \infty} |\tilde{\Psi}_i(x)| = 0 \right\}$$

is closed and nowhere-dense. (In fact, Z has Hausdorff dimension at most one [Tau14, Theorem 1.2].)

2. *There exist $\Psi \in \Gamma(M \backslash Z, \mathrm{Hom}(E, W))$ and a connection A on $\det W|_{M\backslash Z}$ satisfying the limiting equation*

$$\not{D}_A \Psi = 0 \quad \textit{and} \quad \mu(\Psi) = 0 \tag{3.4.1}$$

on $M \backslash Z$ with respect to $\mathbf{p}$. The pointwise norm $|\Psi|$ extends to a Hölder continuous function on all of M and

$$Z = |\Psi|^{-1}(0).$$

Moreover, A is flat with monodromy in $\mathbf{Z}_2$.

3. *On $M \backslash Z$, up to gauge transformations, $\tilde{\Psi}_i$ weakly converges to Ψ in $W^{2,2}_{\mathrm{loc}}$ and A_i weakly converges to A in $W^{1,2}_{\mathrm{loc}}$. There is a constant $\gamma > 0$ such that $|\tilde{\Psi}_i|$ converges to $|\Psi|$ in $C^{0,\gamma}(M)$.*

We expect that the convergence $(\tilde{\Psi}_i, A_i) \to (\Psi, A)$ can be improved to C^∞_{loc} on $M \backslash Z$; cf. [Doa17a, Theorem 1.5]. In Section 3.6, we will see that this is indeed the case if Z is empty.

The following proposition will give us a concrete understanding of solutions to the limiting equation (3.4.1) which are defined on all of M, that is: for which the set Z is empty. It is a special case of the Haydys correspondence discussed in Section 2.5.

Proposition 3.4.2 (cf. [HW15, Appendix A])**.** *If $(\Psi, A) \in \Gamma(\mathrm{Hom}(E, W)) \times \mathscr{A}(\det W)$ is a solution of (3.4.1) and Ψ is nowhere vanishing, then:*

1. $\det W$ *is trivial; in particular, $\mathfrak{w}$ is induced by a spin structure,*
2. *after a gauge transformation we can assume that A is the product connection and there exists a unique spin structure $\mathfrak{s}$ inducing $\mathfrak{w}$ and such that Ψ takes values in* $\mathrm{Re}(E \otimes S_{\mathfrak{s}}) \subset \Gamma(\mathrm{Hom}(E, W))$*. Here $S_{\mathfrak{s}}$ is the spinor bundle of $\mathfrak{s}$.*
3. *Ψ lies in the kernel of $\not{D}^{\mathfrak{s}}_{\mathbf{p}} \colon \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E)) \to \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E))$.*

Moreover, any nowhere vanishing element in $\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}}$ for any spin structure $\mathfrak{s}$ inducing $\mathfrak{w}$ gives rise to a solution of (3.4.1).

Remark 3.4.3. The set of spin structures is a torsor over $H^1(M, \mathbf{Z}_2)$ while the set of spin^c structures is a torsor over $H^2(M, Z)$. If $\beta \colon H^1(M, \mathbf{Z}_2) \to H^2(M, \mathbf{Z})$ denotes the Bockstein homomorphism in the exact sequence

$$\cdots \to H^1(M, \mathbf{Z}_2) \xrightarrow{\beta} H^2(M, \mathbf{Z}) \xrightarrow{2\times} H^2(M, \mathbf{Z}) \to \cdots,$$

then the set of a all spin structures $\mathfrak{s}$ inducing the spin^c structure $\mathfrak{w}$ is a torsor over $\ker \beta$. The set of all spin^c structures $\mathfrak{w}$ with trivial determinant is a torsor over $\ker 2\times$, the 2–torsion subgroup of $H^2(M, \mathbf{Z})$.

Proof of Proposition 3.4.2. Fix a spin structure $\mathfrak{s}_0$ and a Hermitian line bundle L which induce $\mathfrak{w}$; in particular, $W = S_{\mathfrak{s}_0} \otimes L$ and A induces a connection A_0 on L. By Proposition 3.3.1,

$$\mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^2, \mathbf{H}) /\!/\!/ \mathrm{U}(1) = \Big(\mathrm{Re}(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2) \backslash \{0\}\Big) / \mathbf{Z}_2;$$

hence, Ψ gives rise to a section $s \in \Gamma(\mathfrak{X})$ with

$$\mathfrak{X} = (\mathrm{Re}(S_{\mathfrak{s}_0} \otimes E) \backslash \{0\}) / \mathbf{Z}_2$$

satisfying the Fueter equation. In this case it means simply that local lifts of s to $\mathrm{Re}(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2)$ satisfy the Dirac equation. Recall from Section 2.5 that the Haydys correspondence asserts that:

- any $s \in \Gamma(\mathfrak{X})$ can be lifted; that is: there exist a Hermitian line bundle L, $\Psi \in \Gamma(\mathrm{Hom}(E, S_{\mathfrak{s}_0} \otimes L)) = \Gamma(\mathrm{Hom}(E, W))$, as well as $A_0 \in \mathscr{A}(L)$ satisfying (3.4.1), and
- L is determined by s up to isomorphism and any two lifts of s are related by a unique gauge transformation in $\mathscr{G}(L)$.

We claim that s can be, in fact, lifted to a section $\tilde{\Psi} \in \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}_0}) \otimes \mathfrak{l})$ for some Euclidean line bundle $\mathfrak{l}$. To see this, cover M with a finite collection of open balls (U_α) and trivialize $\mathrm{Re}(S_{\mathfrak{s}_0} \otimes E)$ over each U_α. On U_α the section s is given by a smooth function $U_\alpha \to (\mathbf{R}^4 \backslash \{0\}) / \mathbf{Z}_2$ which can be lifted to a map $\tilde{\Psi}_\alpha \colon U_\alpha \to \mathbf{R}^4 \backslash \{0\}$. Over the intersection $U_\alpha \cap U_\beta$ of two different balls U_α, U_β, we have $\tilde{\Psi}_\alpha = f_{\alpha\beta} \tilde{\Psi}_\beta$ for a local constant function $f_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \{-1, +1\}$. The collection $(f_{\alpha\beta})$ is a Čech cocycle with values in $\mathbf{Z}_2$ and defines a $\mathbf{Z}_2$–bundle on M. Let $\mathfrak{l}$ be the associated Euclidean line bundle, The collection of local sections $(\tilde{\Psi}_\alpha)$ defines a section $\tilde{\Psi} \in \Gamma(\mathrm{Re}(S_{\mathfrak{s}_0} \otimes E) \otimes \mathfrak{l})$ as we wanted to show.

Set $\tilde{L} := \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$. By Proposition 3.3.1,

$$\mathrm{Re}(E \otimes S_{\mathfrak{s}_0}) \otimes \mathfrak{l} \subset \mu^{-1}(0) \subset \mathrm{Hom}(E, S_{\mathfrak{s}_0} \otimes \tilde{L}) = \mathrm{Hom}(E, W);$$

and if $\tilde{A} \in \mathscr{A}(\tilde{L})$ denotes the connection induced by the canonical connection on $\mathfrak{l}$, then $\not{D}_{\tilde{A}} \tilde{\Psi} = 0$.

It thus follows from the Haydys correspondence that $L \cong \mathfrak{l} \otimes_{\mathbf{R}} \mathbf{C}$ and, after this identification has been made and a suitable gauge transformation has been applied, $\Psi = \tilde{\Psi}$ and $A_0 = \tilde{A}$. This shows that $\det W = L^2$ is trivial and $\mathfrak{w}$ is induced by the spin structure $\mathfrak{s}$ obtained by twisting $\mathfrak{s}_0$ with $\mathfrak{l}$. ☐

Definition 3.4.4. Denote by $\mathscr{P}^{\mathrm{reg}} \subset \mathscr{P}$ the subset consisting of those $\mathbf{p}$ for which the real Dirac operator, introduced at the end of Section 3.1,

$$\not{D}^{\mathfrak{s}}_{\mathbf{p}} \colon \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E)) \to \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E)) \tag{3.4.2}$$

is invertible for all spin structures $\mathfrak{s}$. Set

$$\mathscr{Q} := \mathscr{P} \times \mathscr{Z},$$

and denote by

$$\mathscr{Q}^{\mathrm{reg}} \subset \mathscr{Q}$$

the subset consisting of those $(\mathbf{p}, \eta)$ for which $\mathbf{p} \in \mathscr{P}^{\mathrm{reg}}$ and every solution (Ψ, A) of (3.2.2) with respect to $(\mathbf{p}, \eta)$ is irreducible and unobstructed.

The next proposition follows from Proposition 3.2.4 and [Ang96, Theorem 1.5; Mai97, Theorem 1.2].

Proposition 3.4.5. *If $b_1(M) > 0$, then $\mathscr{Q}^{\mathrm{reg}}$ is residual in $\mathscr{Q}$.*

Proposition 3.4.6. *If $(\mathbf{p}, \eta) \in \mathscr{Q}^{\mathrm{reg}}$ and there are no singular harmonic $\mathbf{Z}_2$ spinors with respect to $\mathbf{p}$, then $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is compact. In particular, $n_{\mathfrak{w}}(\mathbf{p}, \eta) \in \mathbf{Z}$ as in (3.2.9) is defined.*

Proof. By hypothesis we know that there are no singular harmonic $\mathbf{Z}_2$ spinors. By the definition of $\mathscr{P}^{\mathrm{reg}}$ there are also no harmonic spinors. It thus follows from Theorem 3.4.1 and Proposition 3.4.2 that $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is compact. □

3.5 WALL-CROSSING AND THE SPECTRAL FLOW

In the absence of singular harmonic $\mathbf{Z}_2$ spinors, we can define $n_{\mathfrak{w}}(\mathbf{p}, \eta)$ for every $(\mathbf{p}, \eta) \in \mathscr{Q}^{\mathrm{reg}}$. However, $\mathscr{Q}^{\mathrm{reg}}$ is not path-connected and $n_{\mathfrak{w}}(\mathbf{p}, \eta)$ does depend on the path-connected component of $\mathscr{Q}^{\mathrm{reg}}$ in which $(\mathbf{p}, \eta)$ lies. We study the wall-crossing for $n_{\mathfrak{w}}(\mathbf{p}, \eta)$ by analyzing the family of moduli spaces $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_t, \eta_t)$ along paths of the following kind.

Definition 3.5.1. Given $\mathbf{p}_0, \mathbf{p}_1 \in \mathscr{P}^{\mathrm{reg}}$, denote by $\mathscr{P}^{\mathrm{reg}}(\mathbf{p}_0, \mathbf{p}_1)$ the space of smooth paths from $\mathbf{p}_0$ to $\mathbf{p}_1$ in $\mathscr{P}$ such that for every spin structure $\mathfrak{s}$:

1. the path of Dirac operators $\left(\not{D}^{\mathfrak{s}}_{\mathbf{p}_t}\right)_{t\in[0,1]}$ has transverse spectral flow and
2. whenever the spectrum of $\not{D}^{\mathfrak{s}}_{\mathbf{p}_t}$ crosses zero, $\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_t}$ is spanned by a nowhere vanishing section $\Psi \in \Gamma(\mathrm{Re}(E \otimes S_{\mathfrak{s}}))$.

Given $(\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1) \in \mathscr{Q}^{\mathrm{reg}}$, denote by $\mathscr{Q}^{\mathrm{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1))$ the space of smooth paths $(\mathbf{p}_t, \eta_t)_{t\in[0,1]}$ from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$ in $\mathscr{Q}$ such that (1) and (2) hold and, moreover:

1. For every $t_0 \in [0,1]$, every solution (Ψ, A) of (3.2.2) is irreducible and either it is unobstructed (that is, the linearized operator $L_{\Psi,A}$ is invertible) or else $\operatorname{coker} L_{\Psi,A}$ has dimension one and is spanned by
$$\pi\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_0}\begin{pmatrix} -\not{D}_{\mathbf{p}_t,A}\Psi \\ *(F_A + \eta_t - \mu_{\mathbf{p}_t}(\Psi)) \\ 0 \end{pmatrix}\right)$$
where $\pi\colon \Gamma(\mathrm{Hom}(E, W)) \oplus \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}) \to \operatorname{coker} L_{\Psi,A}$ denotes the L^2–orthogonal projection.
2. For any Ψ as in (2) with $\|\Psi\|_{L^2} = 1$, denote by
$$\left\{\left(\Psi_\varepsilon = \Psi + \varepsilon^2\psi + O(\varepsilon^4), A_\varepsilon; t(\varepsilon) = t_0 + O(\varepsilon^2)\right) : 0 \leqslant \varepsilon \ll 1\right\}$$
the family of solutions to
$$\begin{aligned} \not{D}_{A_\varepsilon,\mathbf{p}_{t(\varepsilon)}}\Psi_\varepsilon &= 0, \\ \varepsilon^2\left(\frac{1}{2}F_{A_\varepsilon} + \eta_{t(\varepsilon)}\right) &= \mu_{\mathbf{p}_{t(\varepsilon)}}(\Psi_\varepsilon), \quad \text{and} \\ \|\Psi_\varepsilon\|_{L^2} &= 1 \end{aligned}$$

obtained from Theorem 2.8.1. Define also

$$\delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0}) := \langle \slashed{D}_{A_0,\mathbf{p}_{t_0}} \psi, \psi \rangle_{L^2}.$$

We require that $\delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0}) \neq 0$ for all t_0 at which the spectrum of $\slashed{D}^{\mathfrak{s}}_{\mathbf{p}_t}$ crosses zero and Ψ is as in (2).

Condition (1) is necessary since the wall-crossing formula will involve the spectral flow of $\slashed{D}^{\mathfrak{s}}_{\mathbf{p}_t}$. In particular, it guarantees that $\dim\ker \slashed{D}^{\mathfrak{s}}_{\mathbf{p}_t} > 0$ for only finitely many $t \in (0,1)$. Condition (2) ensures that the harmonic $\mathbf{Z}_2$ spinors produced by Theorem 3.1.7 are indeed singular. Condition (1) is used to show that the union of all moduli spaces $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_t, \eta_t)$ as t varies from 0 to 1 is an oriented smooth 1–manifold (i.e., a disjoint union of circles and intervals) with oriented boundary $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_1, \eta_1) \cup -\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_0, \eta_0)$.[5] Finally, condition (2) ensures that we can use the local model from Theorem 2.8.1to study the wall-crossing phenomenon.

The following result shows that a generic path from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$ satisfies the conditions in Definition 3.5.1. Its proof is postponed to Section 3.9.

Proposition 3.5.2. *Given $(\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1) \in \mathscr{Q}^{\mathrm{reg}}$, the subspace $\mathscr{Q}^{\mathrm{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1))$ is residual in the space of all smooth paths from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$ in $\mathscr{Q}$.*

The next three sections are occupied with studying the wall crossing along paths in $\mathscr{Q}^{\mathrm{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1))$. In order to state the wall-crossing formula, we need the following preparation.

Proposition 3.5.3. *Denote by $\mathfrak{s}$ a spin structure inducing the spin^c structure $\mathfrak{w}$ and by A the product connection on $\det W$. If Ψ is a nowhere vanishing section of $\mathrm{Re}(S_{\mathfrak{s}} \otimes E)$, then the following hold:*

1. *Let $\mathfrak{a}_\Psi$ be the algebraic operator given by (3.2.4). The map*
$$\tilde{\mathfrak{a}}_\Psi := |\Psi|^{-1}\mathfrak{a}_\Psi \colon\ (T^*M \oplus \underline{\mathbf{R}}) \otimes i\mathbf{R} \to \mathrm{Im}(S_{\mathfrak{s}} \otimes E)$$
is an isometry. Here, $\mathrm{Im}(S_{\mathfrak{s}} \otimes E)$ denotes the imaginary part of $S_{\mathfrak{s}} \otimes E$ defined using the real structure on $S_{\mathfrak{s}} \otimes E$.
2. *Denote by $\slashed{D}_{\mathrm{Im}}$ the restriction of $\slashed{D}_{A,\mathbf{p}}$ to $\mathrm{Im}(S_{\mathfrak{s}} \otimes E) \subset \mathrm{Hom}(E, W)$ and define the operator $\mathfrak{d}_\Psi \colon\ \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R}) \to \Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R})$ by*
$$\mathfrak{d}_\Psi := \tilde{\mathfrak{a}}_\Psi^* \circ \slashed{D}_{\mathrm{Im}} \circ \tilde{\mathfrak{a}}_\Psi.$$
For each $t \in [0,1]$, the operator
$$\mathfrak{d}_\Psi^t := (1-t)\mathfrak{d}_\Psi + t\mathfrak{d}$$
is a self-adjoint and Fredholm.

Proof. The fact that $\tilde{\mathfrak{a}}_\Psi$ is an isometry is a consequence of

$$\mathfrak{a}_\Psi^* \mathfrak{a}_\Psi = |\Psi|^2 \tag{3.5.1}$$

which in turn follows from the following calculation valid for all (a, ξ) and (b, η) in $\Omega^1(M, i\mathbf{R}) \oplus \Omega^0(M, i\mathbf{R})$:

$$\begin{aligned} \langle \mathfrak{a}_\Psi(a,\xi), \mathfrak{a}_\Psi(b,\eta)\rangle &= \langle \bar{\gamma}(a)\Psi + \rho(\xi)\Psi, \bar{\gamma}(b)\Psi + \rho(\eta)\Psi \rangle \\ &= |\Psi|^2(\langle a, b\rangle + \langle \xi, \eta\rangle). \end{aligned}$$

5 Here $-\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_0, \eta_0)$ is the same space as $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_0, \eta_0)$, but all the orientations are reversed.

Since

$$\begin{aligned} \not{D}_{\mathrm{Im}}\mathfrak{a}_\Psi(a,\xi) &= \not{D}_{\mathrm{Im}}(\bar{\gamma}(a)\Psi + \rho(\xi)\Psi) \\ &= \bar{\gamma}(*\mathrm{d}a)\Psi + \rho(\mathrm{d}^*a)\Psi + \bar{\gamma}(\mathrm{d}\xi)\Psi \\ &\quad - \bar{\gamma}(a)\not{D}_{\mathrm{Re}}\Psi + \rho(\xi)\not{D}_{\mathrm{Re}}\Psi - 2\sum_{i=1}^{3}\rho(a(e_i))\nabla_{e_i}\Psi \\ &= \mathfrak{a}_\Psi\mathfrak{d}(a,\xi) - 2\sum_{i=1}^{3}\rho(a(e_i))\nabla_{e_i}\Psi, \end{aligned} \tag{3.5.2}$$

we have

$$\mathfrak{d}_\Psi = \mathfrak{d} + \mathfrak{e}_\Psi$$

with $\mathfrak{e}_\Psi$ a zeroth order operator depending on Ψ and its derivative. This implies that $\mathfrak{d}^t_\Psi$ is a Fredholm operator. By construction $\mathfrak{d}^t_\Psi$ is self-adjoint. □

Definition 3.5.4. In the situation of Proposition 3.5.3, define

$$\sigma(\Psi,\mathbf{p}) := (-1)^{b_1(M)} \cdot (-1)^{\mathrm{SF}\left((-\mathfrak{d}^t_\Psi)_{t\in[0,1]}\right)}.$$

Remark 3.5.5. The operator $\mathfrak{d}_\Psi$ only depends on Ψ up to multiplication by a constant in $\mathbf{R}^*$; hence, the same holds for $\sigma(\Psi,\mathbf{p})$.

Definition 3.5.6. For a pair of nowhere vanishing sections $\Psi,\Phi \in \Gamma(\mathrm{Re}(S_\mathfrak{s}\otimes E))$ we define their *relative degree* $\deg(\Psi,\Phi)\in\mathbf{Z}$ as follows. Choose any trivializations of E and $S_\mathfrak{s}$ compatible with the $\mathrm{SU}(2)$ structures. In the induced trivialization of $\mathrm{Re}(S_\mathfrak{s}\otimes E)$ the sections $\Psi/|\Psi|$ and $\Phi/|\Phi|$ are represented by maps $M\to S^3$. Set

$$\deg(\Psi,\Phi) := \deg(\Psi/|\Psi|) - \deg(\Phi/|\Phi|).$$

This number does not depend on the choice of the trivializations.

Theorem 3.5.7. *Let $(\mathbf{p}_t,\eta_t)_{t\in[0,1]} \in \mathscr{Q}^{\mathrm{reg}}((\mathbf{p}_0,\eta_0),(\mathbf{p}_1,\eta_1))$. For each spin structure $\mathfrak{s}$ inducing the spin^c structure $\mathfrak{w}$, denote*

- *by $\{t^\mathfrak{s}_1,\dots,t^\mathfrak{s}_{N_\mathfrak{s}}\}\subset[0,1]$ the set of times at which the spectrum of $\not{D}^\mathfrak{s}_{\mathbf{p}_t}$ crosses zero*[6]

and, for each $i=1,\dots,N_\mathfrak{s}$, denote

- *by $\chi^\mathfrak{s}_i\in\{\pm1\}$ the sign of the spectral crossing of the family $(\not{D}^\mathfrak{s}_{\mathbf{p}_t})$ at $t^\mathfrak{s}_i$ and*
- *by $\Psi^\mathfrak{s}_i$ a nowhere vanishing spinor spanning* $\ker\not{D}^\mathfrak{s}_{\mathbf{p}_t}$.

If there are no singular harmonic $\mathbf{Z}_2$ spinors with respect to $\mathbf{p}_t$ for any $t\in[0,1]$, then

$$n_\mathfrak{w}(\mathbf{p}_1,\eta_1) = n_\mathfrak{w}(\mathbf{p}_0,\eta_0) + \sum_\mathfrak{s}\sum_{i=1}^{N_\mathfrak{s}}\chi^\mathfrak{s}_i\cdot\sigma(\Psi^\mathfrak{s}_i,\mathbf{p}_{t^\mathfrak{s}_i}) \tag{3.5.3}$$

or, equivalently,

$$n_\mathfrak{w}(\mathbf{p}_1,\eta_1) = n_\mathfrak{w}(\mathbf{p}_0,\eta_0) + \sum_\mathfrak{s}\chi^\mathfrak{s}_1\cdot\sigma(\Psi^\mathfrak{s}_1,\mathbf{p}_{t^\mathfrak{s}_1})\cdot\sum_{i=1}^{N_\mathfrak{s}}(-1)^{i+1}\cdot(-1)^{\deg(\Psi^\mathfrak{s}_1,\Psi^\mathfrak{s}_i)}. \tag{3.5.4}$$

Here the sums are over all spin structures $\mathfrak{s}$ inducing $\mathfrak{w}$.

6 This set is finite by Definition 3.5.1item 1.

Remark 3.5.8. In particular, Theorem 3.5.7 implies that $n_{\mathfrak{w}}(\mathbf{p}, \eta)$ does not depend on η.

The proof of the (3.5.3) proceeds by analyzing the 1–parameter family of moduli spaces

$$\mathfrak{W} := \bigcup_{t\in[0,1]} \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_t, \eta_t).$$

By Definition 3.5.1(1), $\mathfrak{W}$ is an oriented, one-dimensional manifold with oriented boundary

$$\partial\mathfrak{W} = \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_0, \eta_0) \cup -\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_1, \eta_1).$$

If $\mathfrak{W}$ were compact, then it would follow that $n_{\mathfrak{w}}(\mathbf{p}_1, \eta_1) = n_{\mathfrak{w}}(\mathbf{p}_0, \eta_0)$. However, $\mathfrak{W}$ may be non-compact.

3.6 COMPACTIFICATION OF THE COBORDISM

Set

$$\overline{\mathfrak{W}} := \left\{(t, \varepsilon, [\Psi, A]) \in [0,1] \times [0,\infty) \times \frac{\Gamma(\mathrm{Hom}(E, W) \times \mathscr{A}(\det W)}{\mathscr{G}(\det W)} : (*)\right\}$$

with $(*)$ meaning that:

- the differential equation

$$\begin{aligned} \not{D}_{A,\mathbf{p}_t}\Psi &= 0, \\ \varepsilon^2\left(\frac{1}{2}F_A + \eta_t\right) &= \mu_{\mathbf{p}_t}(\Psi), \quad \text{and} \\ \|\Psi\|_{L^2} &= 1 \end{aligned} \tag{3.6.1}$$

  holds and

- if $\varepsilon = 0$, then Ψ is nowhere vanishing.

Equip $\overline{\mathfrak{W}}$ with the C^∞–topology. We have a natural embedding $\mathfrak{W} \hookrightarrow \overline{\mathfrak{W}}$ given by $(t, [\Psi, A]) \mapsto (t, \varepsilon, [\tilde{\Psi}, A])$ with $\varepsilon := 1/\|\Psi\|_{L^2}$ and $\tilde{\Psi} := \Psi/\|\Psi\|_{L^2}$.

Proposition 3.6.1. *$\mathfrak{W}$ is dense in $\overline{\mathfrak{W}}$.*

Proof. If $(t_0, \varepsilon, [\Psi, A]) \in \overline{\mathfrak{W}}\backslash\mathfrak{W}$, then $\varepsilon = 0$. It follows from Definition 3.5.1 and Theorem 2.8.1, that there are is a family $\{(\Psi_\varepsilon, A_\varepsilon; t(\varepsilon)) : 0 \leqslant \varepsilon \ll 1\}$ of solutions to

$$\begin{aligned} \not{D}_{A_\varepsilon,\mathbf{p}_{t(\varepsilon)}}\Psi_\varepsilon &= 0 \quad \text{and} \\ \varepsilon^2\left(\frac{1}{2}F_{A_\varepsilon} + \eta_{t(\varepsilon)}\right) &= \mu_{\mathbf{p}_{t(\varepsilon)}}(\Psi_\varepsilon) \end{aligned}$$

with $(\Psi_\varepsilon, A_\varepsilon)$ converging to (Ψ, A) in C^∞ and $t(\varepsilon)$ converging to t_0 as ε tends to zero. Consequently, $\mathfrak{W}$ is dense in $\overline{\mathfrak{W}}$. ☐

That $\overline{\mathfrak{W}}$ is compact does not follows from Theorem 3.4.1; it does, however, follow from the next result.

Proposition 3.6.2. *Let $(\mathbf{p}_i, \eta_i)$ be a sequence in $\mathscr{P} \times \mathscr{Z}$ which converges to $(\mathbf{p}, \eta)$ in C^∞. Let $(\varepsilon_i, \Psi_i, A_i)$ be a sequence of solutions of*

$$\begin{aligned} \not{D}_{A_i, \mathbf{p}_i} \Psi_i &= 0, \\ \varepsilon^2 \left(\frac{1}{2} F_{A_i} + \eta_i \right) &= \mu_{\mathbf{p}_i}(\Psi_i), \quad \textit{and} \\ \|\Psi_i\|_{L^2} &= 1 \end{aligned} \tag{3.6.2}$$

with $\lim_{i \to \infty} \varepsilon_i = 0$. If the set

$$Z := \left\{ x \in M : \limsup_{i \to \infty} |\Psi_i(x)| = 0 \right\}$$

is empty, then, after passing to a subsequence and applying gauge transformations, (Ψ_i, A_i) converges in C^∞ to a solution $(\Psi, A) \in \Gamma(\mathrm{Hom}(E, W)) \times \mathscr{A}(\det W)$ of

$$\not{D}_{A, \mathbf{p}} \Psi = 0, \quad \mu_{\mathbf{p}}(\Psi) = 0, \quad \textit{and} \quad \|\Psi\|_{L^2} = 1.$$

Proposition 3.6.3. *$\overline{\mathfrak{W}}$ is compact.*

Proof assuming Proposition 3.6.2. We need to show that any sequence $(t_i, \varepsilon_i, [\Psi_i, A_i])$ in $\mathfrak{W}$ has a subsequence which converges in $\overline{\mathfrak{W}}$. If $\liminf \varepsilon_i > 0$, then this a consequence of standard elliptic estimates and Arzelà–Ascoli. It only needs to be pointed out that ε_i cannot tend to infinity, because otherwise there would be a reducible solution to (3.2.2) which is ruled out by Definition 3.5.1(1).

If $\liminf_{i \to \infty} \varepsilon_i = 0$, then Theorem 3.4.1 asserts that a gauge transformed subsequence of (Ψ_i, A_i) converges weakly in $W^{2,2}_{\mathrm{loc}} \times W^{1,2}_{\mathrm{loc}}$ outside Z. If Z is non-empty, then the limit represents a singular harmonic $\mathbf{Z}_2$ spinors. However, by assumption there are no singular harmonic $\mathbf{Z}_2$ spinors; hence, Z is empty and Proposition 3.6.2 asserts that a gauge transformed subsequence of (Ψ_i, A_i) converges in C^∞. □

The proof of Proposition 3.6.2 is somewhat technical and we omit it. It relies on elliptic regularity and a priori estimates on Ψ and the curvature of A, under the assumption that $|\Psi|$ is bounded below by a positive constant; see [DW17c, Section 5] for details.

3.7 ORIENTATION AT INFINITY

Suppose that $(t_0, 0, [\Psi_0, A_0]) \in \overline{\mathfrak{W}} \backslash \mathfrak{W}$ is a boundary point in $\overline{\mathfrak{W}}$. By Proposition 3.4.2, there exists a spin structure $\mathfrak{s}$ inducing the spin^c structure $\mathfrak{w}$ such that $\Psi_0 \in \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E)) \subset \Gamma(\mathrm{Hom}(E, W))$, $\not{D}^{\mathfrak{s}}_{\mathbf{p}_{t_0}} \Psi_0 = 0$, and A_0 is trivial. By Definition 3.5.1(1), there exists a unique solution $\{\Psi_t : |t - t_0| \ll 1\}$ to

$$\not{D}^{\mathfrak{s}}_{\mathbf{p}_t} \Psi_t = \lambda(t) \Psi_t \quad \text{and} \quad \|\Psi_t\|_{L^2} = 1$$

with $\Psi_{t_0} = \Psi_0$. Moreover, λ is a differentiable function of t near t_0 whose derivative, to be denoted by $\dot{\lambda}$, satisfies $\dot{\lambda}(t_0) \neq 0$. In this situation, the proof of Theorem 2.7.1 in Section 2.7 shows that for any choice of $r \in \mathbf{N}$ there exist $\tau \ll 1$ and $\varepsilon_0 \ll 1$, a C^r map

$$\mathrm{ob}\colon (t_0 - \tau, t_0 + \tau) \times [0, \varepsilon_0) \to \mathbf{R},$$

an open neighborhood V of $(t_0, 0, [\Psi_0, A_0]) \in \overline{\mathfrak{W}}$, and a homeomorphism

$$\mathfrak{x}\colon \mathrm{ob}^{-1}(0) \to V$$

such that:

1. $\mathfrak{x}$ commutes with the projection to the t– and ε–coordinates.
2. The restriction $\mathrm{ob}|_{(t_0-\tau, t_0+\tau)\times(0,\varepsilon_0)}$ is smooth and has expansion

$$\begin{aligned} \mathrm{ob}(t,\varepsilon) &= \dot{\lambda}(t_0)\cdot(t-t_0) - \delta\varepsilon^4 + O\left((t-t_0)^2, \varepsilon^6\right), \\ \partial_\varepsilon \mathrm{ob}(t,\varepsilon) &= -4\delta\varepsilon^3 + O\left((t-t_0)^2, \varepsilon^5\right), \end{aligned} \tag{3.7.1}$$

   with $\delta = \delta(\Psi_0, \mathbf{p}_{t_0}, \eta_{t_0})$ as in Definition 3.5.1(2). In particular, since by assumption $\delta \neq 0$, the equation $\mathrm{ob}(t,\varepsilon) = 0$ can be solved for t:

$$t(\varepsilon) = t_0 + \frac{\delta}{\dot{\lambda}(t_0)}\varepsilon^4 + O(\varepsilon^6). \tag{3.7.2}$$

3. If $(t,\varepsilon) \in (t_0-\tau, t_0+\tau)\times(0,\varepsilon_0]$ satisfies $\mathrm{ob}(t,\varepsilon) = 0$ and

$$\mathfrak{x}(t,\varepsilon) = (t, \varepsilon, [\Psi_\varepsilon, A_\varepsilon])$$

   then $[\varepsilon^{-1}\Psi_\varepsilon, A_\varepsilon]$ is a solution of the Seiberg–Witten equation (3.2.2). We will prove below that this solution is unobstructed; that is, the operator

$$L_{\varepsilon^{-1}\Psi_\varepsilon, A_\varepsilon, \mathbf{p}_{t(\varepsilon)}}$$

   is invertible.

It follows from the above that

$$\mathrm{ob}^{-1}(0) = \left\{\left(t(\varepsilon), \varepsilon\right) : \varepsilon \in [0, \varepsilon_0)\right\}.$$

Therefore, $\overline{\mathfrak{W}}$ is a compact, oriented, one–dimensional manifold with boundary, that is: a finite collection of circles and closed intervals. Its oriented boundary is

$$\partial\overline{\mathfrak{W}} = \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_1, \eta_1) - \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_0, \eta_0) \cup \left(\overline{\mathfrak{W}}\backslash\mathfrak{W}\right).$$

It follows from (3.7.2) that, if $\delta/\dot{\lambda}(0) > 0$, then as t passes through t_0 a solution to (3.2.2) is created. If $\delta/\dot{\lambda}(0) < 0$, then as t passes through t_0 a solution to (3.2.2) is annihilated. The solution that is created/annihilated at t_0 contributes $\mathrm{sign}[\varepsilon^{-1}\Psi_\varepsilon, A_\varepsilon]$ to the signed count of solutions. Consequently, for $\tau \ll 1$, we have the local wall-crossing formula

$$n_{\mathfrak{w}}(\mathbf{p}_{t+\tau}, \eta_{t+\tau}) = n_{\mathfrak{w}}(\mathbf{p}_{t_0-\tau}, \eta_{t_0-\tau}) + \mathrm{sign}(\dot{\lambda}(t_0))\,\mathrm{sign}(\delta)\,\mathrm{sign}[\varepsilon^{-1}\Psi_\varepsilon, A_\varepsilon]. \tag{3.7.3}$$

The main result of this section determines the last two factors in the above formula.

Proposition 3.7.1. *In the above situation, for $\varepsilon \ll 1$, the solution $[\varepsilon^{-1}\Psi_\varepsilon, A_\varepsilon]$ is unobstructed and*

$$\mathrm{sign}(\delta)\,\mathrm{sign}[\varepsilon^{-1}\Psi_\varepsilon, A_\varepsilon] = \sigma(\Psi_0, \mathbf{p}_{t_0}) \tag{3.7.4}$$

with $\sigma(\Psi_0, \mathbf{p}_{t_0})$ as in Definition 3.5.4. In particular, the local wall-crossing formula (3.7.3) can be written in the form

$$n_{\mathfrak{w}}(\mathbf{p}_{t_0+\tau}, \eta_{t_0+\tau}) = n_{\mathfrak{w}}(\mathbf{p}_{t_0-\tau}, \eta_{t_0-\tau}) + \mathrm{sign}(\dot{\lambda}(0))\,\sigma(\Psi_0, \mathbf{p}_{t_0}). \tag{3.7.5}$$

The proof of Proposition 3.7.1, which can be found in [DW17c, Section 6], is one of the most technical parts of [DW17c]. We omit it in this thesis, as it is quite involved. The proof consists of several steps. Some of them are rather standard and amount to tracing definitions or calculating various spectral flows. The crucial part is relating the determinant line bundle used in orienting the Seiberg–Witten moduli spaces to the glued Kuranishi model constructed in Section 2.7. We should stress that Proposition 3.7.1 is crucial for proving the existence of singular harmonic $\mathbf{Z}_2$ spinors, as the final argument requires knowing exactly the signs appearing in the wall-crossing formula.

3.8 PROOF OF THE WALL-CROSSING FORMULAE

This section will conclude the proof of Theorem 3.5.7. The local wall-crossing formula (3.7.3) and Proposition 3.7.1 directly imply the wall-crossing formula

$$n_{\mathfrak{w}}(\mathbf{p}_1, \eta_1) = n_{\mathfrak{w}}(\mathbf{p}_0, \eta_0) + \sum_{\mathfrak{s}} \sum_{i=1}^{N_{\mathfrak{s}}} \chi_i^{\mathfrak{s}} \cdot \sigma(\Psi_i^{\mathfrak{s}}, \mathbf{p}_{t_i^{\mathfrak{s}}}). \qquad 3.5.3$$

In order to prove (3.5.4), we need to relate $\sigma(\Psi_0, \mathbf{p}_0)$ to $\sigma(\Psi_1, \mathbf{p}_1)$ for two different nowhere vanishing spinors Ψ_0 and Ψ_1 and two parameters $\mathbf{p}_0$ and $\mathbf{p}_1$.

Proposition 3.8.1. *Let $(\mathbf{p}_t)_{t\in[0,1]}$ be a path in $\mathscr{P}$. If Ψ_0 and Ψ_1 are two nowhere vanishing sections of $\mathrm{Re}(E \otimes S_{\mathfrak{s}})$, then*

$$\sigma(\Psi_1, \mathbf{p}_1) = \sigma(\Psi_0, \mathbf{p}_0) \cdot (-1)^{\mathrm{SF}(-\not{D}^{\mathfrak{s}}_{\mathbf{p}_t})} \cdot (-1)^{\deg(\Psi_0, \Psi_1)}. \qquad (3.8.1)$$

Here $\deg(\Psi_0, \Psi_1)$ *denotes the relative the degree of* Ψ_0 *and* Ψ_1 *as in Definition 3.5.6.*

Proof. Suppose first that $\deg(\Psi_0, \Psi_1) = 0$ so that we can find a path $(\Psi_t)_{t\in[0,1]}$ of nowhere vanishing sections from Ψ_0 to Ψ_1. It follows from Definition 3.5.4 that

$$\sigma(\Psi_1, \mathbf{p}_1) = \sigma(\Psi_0, \mathbf{p}_0) \cdot (-1)^{\mathrm{SF}(-\mathfrak{d}_{\Psi_t})}.$$

The spectral flow along the path of operators $-\mathfrak{d}_{\Psi_t}$ is identical to that of the path of operators

$$-\tilde{\mathfrak{a}}_{\Psi_0} \circ \mathfrak{d}_{\Psi_t} \circ \tilde{\mathfrak{a}}^*_{\Psi_0}. \qquad (3.8.2)$$

Consider the homotopy of paths

$$(t, s) \mapsto -\tilde{\mathfrak{a}}_{\Psi_{st}} \circ \mathfrak{d}_{\Psi_t} \circ \tilde{\mathfrak{a}}^*_{\Psi_{st}}.$$

It is well-defined since Ψ_t is nowhere vanishing for $t \in [0,1]$. For $s = 0$ it gives us the path (3.8.2) whereas for $s = 1$ we obtain

$$-\tilde{\mathfrak{a}}_{\Psi_t} \circ \mathfrak{d}_{\Psi_t} \circ \tilde{\mathfrak{a}}^*_{\Psi_t} = -\not{D}^{\mathfrak{s}}_{\mathbf{p}_t}.$$

Since the spectral flow is a homotopy invariant, we have $\mathrm{SF}(-\mathfrak{d}_{\Psi_t}) = \mathrm{SF}(-\not{D}^{\mathfrak{s}}_{\mathbf{p}_t})$ which proves (3.8.1) if $\deg(\Psi_0, \Psi_1) = 0$.

It remains to deal with the case $\deg(\Psi_0, \Psi_1) \neq 0$. By the above, we can assume that $\mathbf{p}_t = \mathbf{p}$ for all $t \in [0,1]$. Since $\sigma(\Psi_1, \mathbf{p})$ and $\sigma(\Psi_2, \mathbf{p})$ are defined using the spectral flow from $\mathfrak{d}_{\Psi_0}$ and $\mathfrak{d}_{\Psi_1}$ respectively to a given elliptic operator (see Definition 3.5.4), it follows that for any path of elliptic operators $(\mathfrak{d}_t)_{t\in[0,1]}$ connecting $\mathfrak{d}_{\Psi_0}$ and $\mathfrak{d}_{\Psi_1}$ we have

$$\sigma(\Psi_1, \mathbf{p}) \cdot \sigma(\Psi_0, \mathbf{p}) = (-1)^{\mathrm{SF}(-\mathfrak{d}_t)}.$$

By work of Atiyah–Patodi–Singer [APS76, Section 7], the spectral flow $\mathrm{SF}(-\mathfrak{d}_t)$ is equal to the index of an elliptic operator $\mathfrak{D}$ on $S^1 \times M$ constructed as follows. Set $V := (T^*M \oplus \underline{\mathbf{R}}) \otimes i\mathbf{R}$ and define an isometry $f\colon V \to V$ by

$$f := \tilde{\mathfrak{a}}^*_{\Psi_0} \tilde{\mathfrak{a}}_{\Psi_1}$$

By the definition of f, we have

$$\mathfrak{d}_{\Psi_1} = f^{-1} \circ \mathfrak{d}_{\Psi_0} \circ f.$$

Let $\mathbf{V} \to S^1 \times M$ be the vector bundle obtained as the mapping torus of f; that is, $\mathbf{V} = V \times [0,1]/\sim$ where $\sim$ denotes the equivalence relation $(v, 0) \sim (f(v), 1)$. If, as before, $(\mathfrak{d}_t)_{t\in[0,1]}$ is a family of elliptic operators connecting $\mathfrak{d}_{\Psi_0}$ and $\mathfrak{d}_{\Psi_1}$, then the operator $\partial_t - \mathfrak{d}_t$ on the pull-back of V to $[0,1] \times M$, with t denoting the coordinate on $[0,1]$, gives rise to a first order elliptic operator $\mathfrak{D}$ on $\mathbf{V} \to S^1 \times M$ whose index equals $\mathrm{SF}(\mathfrak{d}_t) = -\mathrm{SF}(-\mathfrak{d}_t)$. We compute this index as follows. Under the isomorphism $\tilde{\mathfrak{a}}_{\Psi_0}$ between V and $\mathrm{Re}(S_{\mathfrak{s}} \otimes E)$ the operator $\mathfrak{d}_{\Psi_0}$ corresponds to $\not{D}^{\mathfrak{s}}_{\mathbf{p}}$. Moreover, under this isomorphism, the complex-linear extension of f corresponds to an isomorphism of $S_{\mathfrak{s}} \otimes E$ given by a gauge transformation g of degree $\deg(\Psi_0, \Psi_1)$ of the $\mathrm{SU}(2)$–bundle E (this is because in a local trivialization f is given simply by right-multiplication by a quaternion-valued function). Thus, the complexification of $\mathbf{V}$ is isomorphic to $\mathbf{S}^+_{\mathfrak{s}} \otimes \mathbf{E}$ where $\mathbf{S}^+_{\mathfrak{s}}$ is the positive spinor bundle of $S^1 \times M$ and $\mathbf{E}$ is obtained by gluing $E \to [0,1] \times M$ along $\{0,1\} \times M$ using g. The complexification $\mathfrak{D}^{\mathbf{C}}$ of the operator $\mathfrak{D}$ corresponds in this identification to the Dirac operator on $\mathbf{S}^+_{\mathfrak{s}}$ twisted by a connection on $\mathbf{E}$. By the Atiyah–Singer Index Theorem,

$$\operatorname{index} \mathfrak{D}^{\mathbf{C}} = \int_{S^1\times M} \hat{A}(S^1 \times M)\mathrm{ch}(\mathbf{E}) = -\int_{S^1\times M} c_2(\mathbf{E}) = -\deg(\Psi_0, \Psi_1).$$

The real index of $\mathfrak{D}$ is equal to the complex index of $\mathfrak{D}^{\mathbf{C}}$ and we conclude that

$$\sigma(\Psi_1, \mathbf{p}) \cdot \sigma(\Psi_0, \mathbf{p}) = (-1)^{\mathrm{SF}(-\mathfrak{d}_t)} = (-1)^{\deg(\Psi_0,\Psi_1)}.$$

This completes the proof of this proposition. □

Recall that $t^{\mathfrak{s}}_1, \ldots, t^{\mathfrak{s}}_{N_{\mathfrak{s}}} \in (0,1)$ are the times at which the spectrum of $\not{D}^{\mathfrak{s}}_{\mathbf{p}_t}$ crosses zero. The crossing is transverse with intersection sign $\chi^{\mathfrak{s}}_i \in \{\pm 1\}$. The next result relates $\chi^{\mathfrak{s}}_i$ to $\chi^{\mathfrak{s}}_1$.

Proposition 3.8.2. *For all $i \in \{1, \ldots, N_{\mathfrak{w}}\}$, we have*

$$\chi^{\mathfrak{s}}_i \cdot (-1)^{\mathrm{SF}\left(-\not{D}^{\mathfrak{s}}_{\mathbf{p}_t} : t\in\left[t^{\mathfrak{s}}_1, t^{\mathfrak{s}}_i\right]\right)} = (-1)^{i+1} \cdot \chi^{\mathfrak{s}}_1.$$

Remark 3.8.3. In the above, the operator $-\not{D}^{\mathfrak{s}}_{\mathbf{p}_t}$ is not invertible for $t \in \{t^{\mathfrak{s}}_1, \ldots, t^{\mathfrak{s}}_{N_{\mathfrak{s}}}\}$. According to Remark 3.2.11, the spectral flow from $t^{\mathfrak{s}}_1$ to $t^{\mathfrak{s}}_i$ is defined as the spectral flow of the family $(-\not{D}^{\mathfrak{s}}_{\mathbf{p}_t} + \lambda \mathrm{id})_{t\in[t^{\mathfrak{s}}_1, t^{\mathfrak{s}}_i]}$ for any number $0 < \lambda \ll 1$.

Proof. By induction it suffices to consider the case $i = 2$. The case $i = 2$ can be verified directly case-by-case as follows:

χ_1	χ_2	SF	$\chi_1 \cdot \chi_2 \cdot (-1)^{\mathrm{SF}}$
$+1$	$+1$	-1	-1
$+1$	-1	2	-1
-1	$+1$	0	-1
-1	-1	1	-1

Here $\mathrm{SF} = \mathrm{SF}\big(-\not{D}^{\mathfrak{s}}_{\mathbf{p}_t} : t \in [t^{\mathfrak{s}}_1, t^{\mathfrak{s}}_2]\big)$. □

Combining the two preceding propositions shows that (3.5.3) can equivalently be written as follows:

$$n_{\mathfrak{w}}(\mathbf{p}_1, \eta_1) = n_{\mathfrak{w}}(\mathbf{p}_0, \eta_0) + \sum_{\mathfrak{s}} \chi^{\mathfrak{s}}_1 \cdot \sigma(\Psi^{\mathfrak{s}}_1, \mathbf{p}_{t^{\mathfrak{s}}_1}) \cdot \sum_{i=1}^{N_{\mathfrak{s}}} (-1)^{i+1} \cdot (-1)^{\deg(\Psi^{\mathfrak{s}}_1, \Psi^{\mathfrak{s}}_i)}. \quad 3.5.4$$

This completes the proof of Theorem 3.5.7. □

3.9 TRANSVERSALITY FOR PATHS

The purpose of this section is to prove Proposition 3.5.2.

Proposition 3.9.1. *For any $\mathbf{p}_0, \mathbf{p}_1 \in \mathscr{P}^{\mathrm{reg}}$, the subspace $\mathscr{P}^{\mathrm{reg}}(\mathbf{p}_0, \mathbf{p}_1)$ is residual in the space of all smooth paths from $\mathbf{p}_0$ to $\mathbf{p}_1$ in $\mathscr{P}$.*

Proof. The proof is an application of the Sard–Smale theorem. We will work with Sobolev spaces of sections and connections of class $W^{k,p}$ such that $(k-1)p > 3$. The statement for C^∞ spaces follows then from a standard argument; see, for example, [MS12, Theorem 3.1.5].

Since there are only finitely many spin structures on M, it suffices to consider the conditions (1) and (2) in Definition 3.5.1 for a fixed spin structure $\mathfrak{s}$. Set

$$X := \mathscr{P}(\mathbf{p}_0, \mathbf{p}_1) \times [0,1] \times \frac{\Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E) \backslash \{0\})}{\mathbf{R}^*}$$

$$\text{and} \quad V := \mathscr{P}(\mathbf{p}_0, \mathbf{p}_1) \times [0,1] \times \frac{\Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E) \backslash \{0\}) \times \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E))}{\mathbf{R}^*}.$$

V is a vector bundle over X. Define a section $\sigma \in \Gamma(V)$ by

$$\sigma\Big((\mathbf{p}_t)_{t\in[0,1]}, t_*, [\Psi]\Big) := \Big((\mathbf{p}_t)_{t\in[0,1]}, t_*, [(\Psi, \not{D}_{\mathbf{p}_{t_*}}\Psi)]\Big).$$

We can identify a neighborhood of $[\Psi] \in \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E) \backslash \{0\})/\mathbf{R}^*$ with the L^2–orthogonal complement $\Psi^\perp \subset \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E)$. This gives us a local trivialization of V in which σ can be identified with the map

$$\sigma\Big((\mathbf{p}_t)_{t\in[0,1]}, t_*, \psi\Big) = \Big((\mathbf{p}_t)_{t\in[0,1]}, t_*, \not{D}_{\mathbf{p}_{t_*}}(\Psi + \psi)\Big) \quad \text{for all } \psi \in \Psi^\perp.$$

In particular, for a fixed path $(\mathbf{p}_t)_{t\in[0,1]}$, the map $\sigma((\mathbf{p}_t)_{t\in[0,1]}, \cdot)$ defines a Fredholm section of index zero, since $\psi \mapsto \not{D}_{\mathbf{p}_t}(\Psi + \psi)$ has index -1 and $\dim[0,1] = 1$.

Let $\mathbf{x} := \Big((\mathbf{p}_t)_{t\in[0,1]}, t_*, [\Psi]\Big) \in X$ and denote by $\mathrm{d}_{\mathbf{x}}\sigma$ the linearization of σ at $\mathbf{x}$ (and computed in the above trivialization). We will prove that $\mathrm{d}_{\mathbf{x}}\sigma$ is surjective provided $\sigma(\mathbf{x}) = 0$. If $\Phi \in V_{\mathbf{x}} = \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E))$ is orthogonal to the image of $\mathrm{d}_{\mathbf{x}}\sigma$, then it follows that

$$\langle \bar{\gamma}(b)\Psi, \Phi\rangle_{L^2} = 0 \quad \text{for all } b \in \Omega^1(M, \mathfrak{su}(E)). \tag{3.9.1}$$

Since Ψ is harmonic, its zero set must be nowhere dense. Clifford multiplication by $T^*M \otimes \mathfrak{su}(E)$ on $\mathrm{Re}(S_{\mathfrak{s}} \otimes E)$ induces a isomorphism between $T^*M \otimes \mathfrak{su}(E)$ and trace-free symmetric endomorphisms of $\mathrm{Re}(S_{\mathfrak{s}} \otimes E)$. With this in mind it follows from (3.9.1) that $\Psi = 0$. This proves that $\mathrm{d}_{\mathbf{x}}\sigma$ is surjective.

It follows that $\sigma^{-1}(0)$ is a smooth submanifold of X and the projection map $\pi\colon \sigma^{-1}(0) \to \mathscr{P}(\mathbf{p}_0, \mathbf{p}_1)$ is a Fredholm map of index zero. The kernel of $\mathrm{d}\pi$ at $\mathbf{x} \in \sigma^{-1}(0)$ can be identified with the kernel of the linearization of σ in the directions of $[0,1]$ and $\Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E)\backslash\{0\})/\mathbf{R}^*$. Writing down this linearization explicitly, we see that the condition $\ker \mathrm{d}\pi(\mathbf{x}) = \{0\}$ implies that Ψ spans $\ker \not{D}_{\mathbf{p}_{t_*}}$ and t_* is a regular crossing of the spectral flow of $(\not{D}_{\mathbf{p}_t})$. On the other hand, since π is a Fredholm map of index zero, $\ker d\pi(\mathbf{x}) = \{0\}$ is equivalent to $\mathbf{x}$ being a regular point of π. By the Sard–Smale theorem, the subspace of regular values of π is residual; hence, the set of those $(\mathbf{p}_t)_{t\in[0,1]}$ in $\mathscr{P}(\mathbf{p}_0, \mathbf{p}_1)$ for which the condition (1) in Definition 3.5.1 holds is residual.

To deal with condition (2) in Definition 3.5.1, we consider the vector bundle

$$W := \mathscr{P}(\mathbf{p}_0, \mathbf{p}_1) \times [0,1] \times \frac{\Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E)\backslash\{0\}) \times \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E)) \times \mathrm{Re}(S_{\mathfrak{s}} \otimes E)}{\mathbf{R}^*}$$

over $X \times M$ and define a section $\boldsymbol{\tau} \in \Gamma(W)$ by

$$\boldsymbol{\tau}\Big((\mathbf{p}_t)_{t\in[0,1]}, t_*, [\Psi], y\Big) := \Big((\mathbf{p}_t)_{t\in[0,1]}, t_*, [(\Psi, \not{D}_{\mathbf{p}_{t_*}}\Psi, \Psi(y))]\Big).$$

For a fixed path $(\mathbf{p}_t)_{t\in[0,1]}$, the map $\boldsymbol{\tau}((\mathbf{p}_t)_{t\in[0,1]}, \cdot)$ defines a Fredholm section of index -1. Note that for $\mathbf{x} := \Big((\mathbf{p}_t)_{t\in[0,1]}, t_*, [\Psi]\Big) \in X$ and $y \in M$ the condition $\boldsymbol{\tau}((\mathbf{p}_t)_{t\in[0,1]}, \mathbf{x}, y) = 0$ is equivalent to $\not{D}_{\mathbf{p}_{t_*}}\Psi = 0$ and $\Psi(y) = 0$. We prove that the linearization of $\boldsymbol{\tau}$ is surjective at any $(\mathbf{x}, y)$ satisfying these equations. If $(\Phi, \phi) \in W_{(\mathbf{x},y)} = \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E)) \times \mathrm{Re}(S_{\mathfrak{s}} \otimes E)_y$ is orthogonal to the image of $(\mathrm{d}\tau)_{(\mathbf{x},y)}$, then (3.9.1) holds and, moreover,

$$\langle \not{D}_{\mathbf{p}_{t_*}}(\Psi + \psi), \Phi\rangle_{L^2} + \langle \psi(x), \phi\rangle = 0 \quad \text{for all } \psi \in \Psi^\perp \tag{3.9.2}$$

Since $\not{D}_{\mathbf{p}_{t_*}}\Psi = 0$ and $\Psi(y) = 0$, (3.9.2) holds in fact for all $\psi \in \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E))$ and we conclude that $\not{D}_{\mathbf{p}_{t_*}}\Psi = 0$. Plugging this back into (3.9.2) yields $\langle \psi(x), \phi\rangle = 0$ for all ψ, which implies that $\phi = 0$. As before (3.9.1) implies that $\Phi = 0$.

It follows that $\boldsymbol{\tau}^{-1}(0)$ is smooth and the projection $\rho\colon \boldsymbol{\tau}^{-1}(0) \to \mathscr{P}(\mathbf{p}_0, \mathbf{p}_1)$ is a Fredholm map of index -1; in particular, the preimage of a regular value must be empty. It follows that the paths $(\mathbf{p}_t)_{t\in[0,1]} \in \mathscr{P}(\mathbf{p}_0, \mathbf{p}_1)$ for which condition (2) in Definition 3.5.1 holds is residual. □

To address condition (2) in Definition 3.5.1 we compute $\delta(\Psi, \mathbf{p}, \eta)$.

Proposition 3.9.2. *Let $(\mathbf{p}_t)_{t\in[0,1]} \in \mathscr{P}(\mathbf{p}_0, \mathbf{p}_1)$, let $(\eta_t)_{t\in[0,1]}$ be a path in $\mathscr{X}$, let $t_0 \in (0,1)$, and let Ψ be a nowhere vanishing section of $\mathrm{Re}(S_{\mathfrak{s}} \otimes E)$ spanning $\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_{t_0}}$ and satisfying $\|\Psi\|_{L^2} = 1$. There is a linear algebraic operator*

$$\mathfrak{f}_{\mathbf{p}_{t_0},\Psi}\colon\ \Omega^2(M, i\mathbf{R}) \to \Omega^2(M, i\mathbf{R})$$

such that

$$\delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0}) = \int_M |\Psi|^{-2} \langle \mathrm{d} * \eta_{t_0} + \mathfrak{f}_{\Psi, \mathbf{p}_{t_0}} \eta_{t_0}, \eta_{t_0} \rangle \tag{3.9.3}$$

with $\delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0})$ *is as in Definition 3.5.1(2).*

Proof. If we denote by

$$\left\{ \left(\Psi_\varepsilon = \Psi + \varepsilon^2 \psi + O(\varepsilon^4), A_\varepsilon; t(\varepsilon) = t_0 + O(\varepsilon^2) \right) : 0 \leqslant \varepsilon \ll 1 \right\}$$

the family of solutions to

$$\begin{aligned} \not{D}_{A_\varepsilon, \mathbf{p}_{t(\varepsilon)}} \Psi_\varepsilon &= 0, \\ \varepsilon^2 \left(\frac{1}{2} F_{A_\varepsilon} + \eta_{t(\varepsilon)} \right) &= \mu_{\mathbf{p}_{t(\varepsilon)}}(\Psi_\varepsilon), \quad \text{and} \\ \|\Psi_\varepsilon\|_{L^2} &= 1 \end{aligned}$$

obtained from Theorem 2.8.1, then

$$\delta = \delta(\Psi, \mathbf{p}_{t_0}, \eta_{t_0}) = \langle \not{D}_{A_0, \mathbf{p}_{t_0}} \psi, \psi \rangle_{L^2}.$$

The connection $A = A_0$ corresponding to Ψ is flat; see Proposition 3.4.2. For the unperturbed equation we would have $t(\varepsilon) = 0$. Since we consider the perturbed equation, however, η enters into the computation of δ. More precisely, by (2.8.1) and (2.8.2), we have

$$\psi = -\not{D}_{\mathrm{Re}}^{-1} \gamma \mathrm{II}^* \nu - \nu \quad \text{with} \quad \nu := (\mathfrak{a}_\Psi^*)^{-1} * \eta.$$

By (3.5.1), we have

$$\nu = |\Psi|^{-2} \bar{\gamma}(*\eta)\Psi.$$

Denote by π_{Re} the projection onto $\mathrm{Re}(S_{\mathfrak{s}} \otimes E)$. From Proposition 2.5.13 we know that for any $a \in \Omega^1(M, i\mathbf{R})$

$$-\gamma \mathrm{II}^* \bar{\gamma}(a) \Psi = \sum_{i=1}^{3} \pi_{\mathrm{Re}} \left(\rho(a(e_i)) \nabla_{e_i}^{A} \Psi \right) = 0.$$

It follows that

$$\psi = -\nu = |\Psi|^{-2} \bar{\gamma}(*\eta)\Psi.$$

Set

$$a := |\Psi|^{-2} * \eta.$$

Since

$$\not{D}_A \bar{\gamma}(a) \Psi = \bar{\gamma}(*\mathrm{d}a)\Psi + \rho(\mathrm{d}^* a)\Psi - 2 \sum_{i=1}^{3} \rho(a_i) \nabla_i \Psi,$$

we have

$$\begin{aligned}\delta &= \int_M \langle \not{D}_A \bar{\gamma}(a)\Psi, \bar{\gamma}(a)\Psi\rangle \\ &= \int_M \langle \bar{\gamma}(*\mathrm{d}a)\Psi, \bar{\gamma}(a)\Psi\rangle + \langle \rho(\mathrm{d}^*a)\Psi, \bar{\gamma}(a)\Psi\rangle \\ &\qquad - 2\sum_{i=1}^{3}\langle \rho(a_i)\nabla_i\Psi, \bar{\gamma}(a)\Psi\rangle.\end{aligned}$$

The first term in the integral is

$$|\Psi|^2\langle *\mathrm{d}a, a\rangle = \langle \mathrm{d} * \left(|\Psi|^{-2}\eta\right), \eta\rangle.$$

The second term vanishes. Therefore,

$$\delta = \int_M |\Psi|^{-2}\langle \mathrm{d} * \eta, \eta\rangle + |\Psi|^{-2}\langle \mathfrak{f}_1\eta, \mathfrak{f}_2\eta\rangle$$

for linear operators $\mathfrak{f}_1, \mathfrak{f}_2\colon\ \Omega^2(M, i\mathbf{R}) \to \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E))$ of order zero. Set $\mathfrak{f}_{\Psi,\mathbf{p}} = \mathfrak{f}_2^*\mathfrak{f}_1$. □

Proposition 3.9.3. *For $(\Psi, \mathbf{p}, \eta) \in \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E)) \times \mathscr{P} \times \mathscr{Z}$ such that Ψ is nowhere vanishing, define $\delta(\Psi, \mathbf{p}, \eta)$ by formula (3.9.3). For each $(\Psi, \mathbf{p})$, the set*

$$\mathscr{Z}^{\mathrm{reg}}(\Psi, \mathbf{p}) = \{\eta \in \mathscr{Z} : \delta(\Psi, \mathbf{p}, \eta) \neq 0\}$$

is open and dense in $\mathscr{Z}$.

Proof. Replace all the spaces in question by their completions with respect to the $W^{k,p}$ norm for any k and p satisfying $(k-1)p > 3$. We will prove the statement with respect to the Sobolev topology; the corresponding statement for C^∞ spaces follows then from the Sobolev embedding theorem and the fact that δ is continuous with respect to any of these topologies.

By Proposition 3.9.2,

$$(\mathrm{d}\delta)_\eta[\hat{\eta}] = \int_M |\Psi|^{-2}\langle 2\mathrm{d} * \eta + \bar{\mathfrak{f}}_{\Psi,\mathbf{p}}\eta, \hat{\eta}\rangle,$$

where

$$\bar{\mathfrak{f}}_{\Psi,\mathbf{p}}\eta = (\mathfrak{f}_{\Psi,\mathbf{p}} + \mathfrak{f}^*_{\Psi,\mathbf{p}})\eta - 2\mathrm{d}(\log|\Psi|) \wedge *\eta$$

is a linear algebraic operator. Thus, the derivative of δ vanishes along the set $\mathscr{Z}_{crit}(\Psi, \mathbf{p})$ of solutions η to the linear elliptic differential equation

$$\begin{aligned}\mathrm{d}\eta &= 0, \\ \mathrm{d} * \eta + *\bar{\mathfrak{f}}_{\Psi,\mathbf{p}}\eta &= 0.\end{aligned}$$

$\mathscr{Z}_{crit}(\Psi, \mathbf{p})$ a closed, finite-dimensional subspace of $\mathscr{Z}$. By the Implicit Function Theorem, away from $\mathscr{Z}_{crit}(\Psi, \mathbf{p})$, the zero set of $\delta(\Psi, \mathbf{p}, \cdot)$ is a codimension one Banach submanifold of (the Sobolev completion of) $\mathscr{Z}$. Hence, the set

$$\mathscr{Z}^{\mathrm{reg}}(\Psi, \eta) \cap (\mathscr{Z} \backslash \mathscr{Z}_{crit}(\Psi, \eta))$$

is dense. Since δ is continuous, $\mathscr{Z}^{\mathrm{reg}}(\Psi, \eta)$ is open. □

Proof of Proposition 3.5.2. Let $\mathscr{Q}_3$ be the subspace of paths from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$ satisfying Definition 3.5.1(1). The proof of Proposition 3.2.6 shows that $\mathscr{Q}_3$ is residual.

Denote by $\mathscr{Q}_{1,2}$ the space of paths from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$ satisfying conditions (1) and (2) in Definition 3.5.1. By Proposition 3.9.1, $\mathscr{Q}_{1,2}$ is residual in the space of paths from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$. Let $\mathscr{Q}_{1,2,4} \subset \mathscr{Q}_{1,2}$ be the space of paths from $(\mathbf{p}_0, \eta_0)$ to $(\mathbf{p}_1, \eta_1)$ also satisfying Definition 3.5.1(2). Elementary arguments show that $\mathscr{Q}_{1,2,4}$ is open in $\mathscr{Q}_{1,2}$ and we will shortly prove that $\mathscr{Q}_{1,2,4}$ is dense in $\mathscr{Q}_{1,2}$. A set which is open and dense in a residual set is itself residual. It follows that $\mathscr{Q}_{1,2,4}$ is residual; hence, so is $\mathscr{Q}^{\mathrm{reg}}\big((\mathbf{p}_0, \eta_0), (\mathbf{p}_1, \eta_1)\big) = \mathscr{Q}_{1,2,4} \cap \mathscr{Q}_3$.

To prove that $\mathscr{Q}_{1,2,4}$ is dense in $\mathscr{Q}_{1,2}$, suppose that $(\mathbf{p}_t, \eta_t)_{t\in[0,1]} \in \mathscr{Q}_{1,2}$. There are finitely many times $0 < t_1 < \ldots < t_n < 1$ for which the kernel of $\not{D}_{\mathbf{p}_{t_i}}$ is non-trivial. For $i = 1, \ldots, n$, denote by Ψ_i a section spanning $\ker \not{D}_{\mathbf{p}_{t_i}}$. By Proposition 3.9.3, for any $\sigma > 0$, there are closed forms $\alpha_1, \ldots, \alpha_n$ such that

$$\delta(\Psi_i, \mathbf{p}_{t_i}, \eta_{t_i} + \alpha_i) \neq 0 \quad \text{and} \quad \|\alpha_i\|_{L^\infty} \leqslant \sigma$$

for every $i = 1, \ldots, n$. Let $(\alpha_t)_{t\in[0,1]}$ be a path of closed forms such that $\alpha_{t_i} = \alpha_i$ for $i = 1, \ldots, n$ and $\|\alpha_t\|_{L^\infty} \leqslant \sigma$ for all $t \in [0,1]$. The path $(\mathbf{p}_t, \eta_t + \alpha_t)_{t\in[0,1]}$ satisfies conditions (1), and (2) in Definition 3.5.1 because these only depend on $(\mathbf{p}_t)_{t\in[0,1]}$. It also satisfies (2) by construction. We conclude that $(\mathbf{p}_t, \eta_t + \alpha_t)_{t\in[0,1]} \in \mathscr{Q}_{1,2,4}$. Since σ is arbitrary, it follows that $\mathscr{Q}_{1,2,4}$ is dense in $\mathscr{Q}_{1,2}$. □

3.10 PROOF OF THE EXISTENCE OF SINGULAR HARMONIC $\mathbf{Z}_2$ SPINORS

In this section we prove Theorem 3.1.7. We begin with defining the set $\mathscr{W}_b$ appearing in its statement.

Definition 3.10.1. Given a spin structure $\mathfrak{s}$, set

$$\begin{aligned}
\mathscr{W}^{\mathfrak{s}} &:= \left\{\mathbf{p} \in \mathscr{P} : \dim\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} > 0\right\},\\
\mathscr{W}^{\mathfrak{s}}_{1,\varnothing} &:= \left\{\mathbf{p} \in \mathscr{P} : \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} = \mathbf{R}\langle\Psi\rangle \text{ with } \Psi \text{ nowhere vanishing}\right\},\\
\mathscr{W}^{\mathfrak{s}}_{1,\star} &:= \left\{\mathbf{p} \in \mathscr{P} : \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} = \mathbf{R}\langle\Psi\rangle \text{ and } \Psi \text{ has a single non-degenerate zero}\right\} \quad \text{and}\\
\mathscr{W}^{\mathfrak{s}}_{b} &:= \mathscr{W}^{\mathfrak{s}} \backslash \mathscr{W}^{\mathfrak{s}}_{1,\varnothing}.
\end{aligned}$$

A zero $x \in \Psi^{-1}(0)$ is non-degenerate if the linear map $(\nabla\Psi)_x \colon T_xM \to \mathrm{Re}(S \otimes E)_x$ has maximal rank (that is, rank three). Set

$$\mathscr{W} := \bigcup_{\mathfrak{s}} \mathscr{W}^{\mathfrak{s}}, \quad \mathscr{W}_b := \bigcup_{\mathfrak{s}} \mathscr{W}^{\mathfrak{s}}_b, \quad \text{and} \quad \mathscr{W}_{1,\varnothing} := \bigcup_{\mathfrak{s}} \mathscr{W}^{\mathfrak{s}}_{1,\varnothing} \backslash \mathscr{W}_b.$$

Here the union is taken over all spin structures $\mathfrak{s}$.

Proposition 3.10.2. *$\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ is a closed, codimension one submanifold of $\mathscr{P}\backslash\mathscr{W}^{\mathfrak{s}}_b$. It carries a coorientation such that the following holds. Let $(\mathbf{p}_t)$ be a path in $\mathscr{P}\backslash\mathscr{W}^{\mathfrak{s}}_b$ with $\mathbf{p}_0, \mathbf{p}_1 \in \mathscr{P}\backslash\mathscr{W}^{\mathfrak{s}}$ which intersects $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ transversely. Denote*

- *by $\{t_1, \ldots, t_N\} \subset (0,1)$ the finite set of times at which the spectrum of $\not{D}^{\mathfrak{s}}_{\mathbf{p}_t}$ crosses zero, i.e., $\mathbf{p}_t \in \mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$*

and, for each $i = 1, \ldots, N$, denote

- *by* $\chi_i \in \{\pm 1\}$ *the sign of the spectral crossing at* t_i *and*
- *by* Ψ_i *a nowhere vanishing spinor spanning* $\ker \not{D}_{\mathbf{p}_t}$.

The intersection number of $(\mathbf{p}_t)$ *with* $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ *is*

$$\sum_{i=1}^{N} \chi_i \cdot \sigma(\Psi_i, \mathbf{p}_{t_i}).$$

Proof. Let $\mathbf{p}_0 \in \mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$. Let $\Psi_0 \in \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_0}$ be such that $\|\Psi_0\|_{L^2} = 1$. It follows from the Implicit Function Theorem that, for some open neighborhood U of $\mathbf{p}_0 \in \mathscr{P}$, there is a unique smooth map $U \to \mathbf{R} \times \Gamma(\mathrm{Re}(S \otimes E))$,

$$\mathbf{p} \mapsto (\lambda(\mathbf{p}), \Psi_{\mathbf{p}})$$

such that

$$\lambda(\mathbf{p}_0) = 0 \quad \text{and} \quad \Psi_{\mathbf{p}_0} = \Psi_0$$

as well as

$$\not{D}^{\mathfrak{s}}_{\mathbf{p}} \Psi_{\mathbf{p}} = \lambda(\mathbf{p}) \Psi_{\mathbf{p}} \quad \text{and} \quad \|\Psi_0\|_{L^2} = 1. \tag{3.10.1}$$

It follows from the Implicit Function Theorem and the openness of the non-vanishing condition that for U sufficiently small, we have

$$U \cap \mathscr{W}^{\mathfrak{s}}_{1,\varnothing} = \lambda^{-1}(0).$$

Since

$$(\mathrm{d}_{\mathbf{p}_0} \lambda)(0, b) = \langle \bar{\gamma}(b) \Psi, \Psi \rangle_{L^2} \tag{3.10.2}$$

and Clifford multiplication induces an isomorphism from $T^*M \otimes \mathfrak{su}(E)$ to trace-free symmetric endomorphisms of $\mathrm{Re}(S \otimes E)$, λ is a submersion provided U is sufficiently small. Hence, $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ is a codimension one submanifold. To see that $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ is closed, observe that $(\mathbf{p}_i)$ is a sequence on $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ with $\mathbf{p}_i \to \mathbf{p} \in \mathscr{P}$, then $\mathbf{p} \in \mathscr{W}^{\mathfrak{s}}$ and thus either in $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ or $\mathscr{W}^{\mathfrak{s}}_{b}$.

The above argument goes through with

$$\mathscr{W}^{\mathfrak{s}}_{1} = \{\mathbf{p} \in \mathscr{P} : \dim \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} = 1\}$$

instead of $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$. Define a coorientation of $\mathscr{W}^{\mathfrak{s}}_{1}$ by demanding that the isomorphism

$$\mathrm{d}_{\mathbf{p}} \lambda \colon \ T_{\mathbf{p}} \mathscr{P} / T_{\mathbf{p}} \mathscr{W}^{\mathfrak{s}}_{1} \to \mathbf{R}$$

is orientation-preserving. This coorientation has the following property. If $(\mathbf{p}_t)_{t \in [0,1]}$ is a path in $\mathscr{P}$ such that $\dim \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_t} \leqslant 1$, $\dim \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_t} = 0$ for $t = 0, 1$, then the intersection number of $(\mathbf{p}_t)_{t \in [0,1]}$ with $\mathscr{W}^{\mathfrak{s}}_{1}$ is precisely the spectral flow of $\not{D}^{\mathfrak{s}}_{\mathbf{p}_t}$. Therefore, we call this coorientation the *spectral coorientation*. $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ is an open subset of $\mathscr{W}^{\mathfrak{s}}_{1}$. Thus it inherits the spectral coorientation; however, this coorientation does not have the desired property.

If $\mathbf{p} \in \mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ and $\Psi_{\mathbf{p}}$ spans $\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}}$, then $\Psi_{\mathbf{p}}$ is nowhere vanishing and Definition 3.5.4 defines $\sigma(\Psi_{\mathbf{p}}, \mathbf{p}) \in \{\pm 1\}$. By Proposition 3.8.1, the $\sigma(\Psi_{\mathbf{p}}, \mathbf{p})$ depends only on $\mathbf{p} \in \mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$; moreover, $\mathbf{p} \mapsto \sigma(\Psi_{\mathbf{p}}, \mathbf{p})$ is locally constant on $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$. The *twisted spectral coorientation* on $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ is defined by demanding that the isomorphism

$$\sigma(\Psi_{\mathbf{p}}, \mathbf{p}) \cdot \mathrm{d}_{\mathbf{p}}\lambda \colon T_{\mathbf{p}}\mathscr{P}/T_{\mathbf{p}}\mathscr{W}^{\mathfrak{s}}_{1,\varnothing} \to \mathbf{R}$$

is orientation-preserving. By definition, for any path $(\mathbf{p}_t)$ as in the statement of the proposition, the intersection number of $(\mathbf{p}_t)$ with $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ with respect to the twisted spectral coorientation is

$$\sum_{i=1}^{N} \chi_i \cdot \sigma(\Psi_i, \mathbf{p}_{t_i}). \qquad \square$$

Theorem 3.10.3. *In the above situation, the following hold.*

1. *The cohomology class $\omega \in H^1(\mathscr{P}\backslash\mathscr{W}_b, \mathbf{Z}) = \mathrm{Hom}(\pi_1(\mathscr{P}\backslash\mathscr{W}_b), \mathbf{Z})$ defined by $\mathscr{W}_{1,\varnothing}$ together with the coorientation from Proposition 3.10.2 is non-trivial.*
2. *If $(\mathbf{p}_0, \eta_0) \in \mathscr{Q}^{\mathrm{reg}}$ and $(\mathbf{p}_t, \eta_t)$ is a loop in $\mathscr{Q}^{\mathrm{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_0, \eta_0))$, then $(\mathbf{p}_t)$ is a path in $\mathscr{P}\backslash\mathscr{W}_b$ and if $\omega([\mathbf{p}_t]) \neq 0$, then there is exists a harmonic $\mathbf{Z}_2$ spinor with respect to some $\mathbf{p}_t$.*

Proof of Theorem 3.1.7 assuming Theorem 3.10.3. The union of the projections of the subsets $\mathscr{Q}^{\mathrm{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_0, \eta_0))$ to $\mathscr{P}(\mathbf{p}_0, \mathbf{p}_0)$, as $(\mathbf{p}_0, \eta_0)$ ranges over $\mathscr{Q}^{\mathrm{reg}}$, is a residual subset of the space of all loops in $\mathscr{P}$. This shows that the loops in $\mathscr{P}\backslash\mathscr{W}_b$ which have a lift to $\mathscr{Q}^{\mathrm{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_0, \eta_0))$ are generic among all loops in $\mathscr{P}\backslash\mathscr{W}_b$. For such loops Theorem 3.10.3 applies and thus Theorem 3.1.7 follows. $\square$

The idea of the proof that $\omega \neq 0$ is to exhibit a loop $(\mathbf{p}_t)$ in $\mathscr{P}\backslash\mathscr{W}_b$ on which ω evaluates non-trivially. More precisely, we will construct such a loop which intersects $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ in two points as illustrated in Figure 3.1, which cannot be joined by a path in $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$; however, they are joined by a path in $\mathscr{W}^{\mathfrak{s}}_{1}$ passing through $\mathscr{W}^{\mathfrak{s}}_{1,\star}$ in a unique point.

While the coorientation on $\mathscr{W}^{\mathfrak{s}}_{1}$ is preserved along this path, the one on $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ is not. Consequently, the intersection number of the loop with $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ is ± 2. The above situation can be arranged so that $(\mathbf{p}_t)$ does not intersect $\mathscr{W}^{\tilde{\mathfrak{s}}}_{1,\varnothing}$ for any other spin structure $\tilde{\mathfrak{s}}$. It follows that

$$\omega([\mathbf{p}_t]) \pm 2 \neq 0.$$

If $(\mathbf{p}_0, \eta_0) \in \mathscr{Q}^{\mathrm{reg}}$ and $(\mathbf{p}_t, \eta_t)$ is a loop in $\mathscr{Q}^{\mathrm{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_0, \eta_0))$ and $\omega([\mathbf{p}_t]) \neq 0$, then there is exists a singular harmonic $\mathbf{Z}_2$ harmonic spinor with respect to some $\mathbf{p}_t$ for otherwise

$$\omega([\mathbf{p}_t]) = 0$$

by Theorem 3.5.7.

Remark 3.10.4. This and the work of Takahashi [Tak15; Tak17] indicate the presence of a wall $\mathscr{W}_{\mathbf{Z}_2} \subset \mathscr{P}$ caused by singular harmonic $\mathbf{Z}_2$ spinors as depicted in Figure 3.1. In light of the above discussion it is a tantalizing question to ask:

> Can the harmonic $\mathbf{Z}_2$ spinors, whose abstract existence is guaranteed by Theorem 3.10.3, be constructed more directly by a gluing construction?

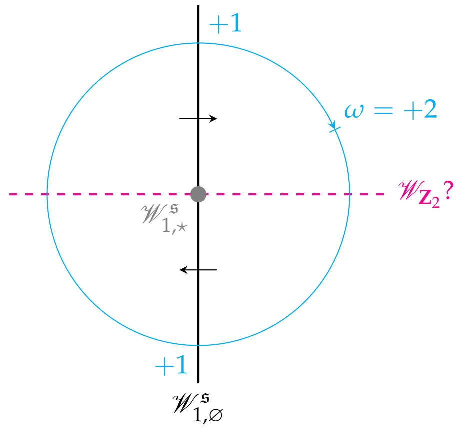


Figure 3.1: A loop linking $\mathscr{W}^{\mathfrak{s}}_{1,\star}$ and pairing non-trivially with ω.

We plan to investigate this problem in future work.

Proof of Theorem 3.10.3. To prove (2), note that if $(\mathbf{p}_0, \eta_0) \in \mathscr{Q}^{\mathrm{reg}}$ and $(\mathbf{p}_t, \eta_t)$ is a loop in $\mathscr{Q}^{\mathrm{reg}}((\mathbf{p}_0, \eta_0), (\mathbf{p}_0, \eta_0))$ and $\omega([\mathbf{p}_t]) \neq 0$, then there is exists a singular harmonic $\mathbf{Z}_2$ spinor with respect to some $\mathbf{p}_t$ for otherwise

$$n(\mathbf{p}_0, \eta_0) = n(\mathbf{p}_0, \eta_0) + \omega([\mathbf{p}_t])$$

by Theorem 3.5.7. Here

$$n(\mathbf{p}, \eta) = \sum_{\mathfrak{w}} n_{\mathfrak{w}}(\mathbf{p}, \eta)$$

and we sum over all spin^c structures $\mathfrak{w}$ with trivial determinant.

In order to prove (1) we will produce a loop pairing non-trivially with ω. The existence of such a loop is ensured by the following result provided we can exhibit a point $\mathbf{p}_\star \in \mathscr{W}^{\mathfrak{s}}_{1,\star}$.

Proposition 3.10.5. *Given $\mathbf{p}_\star \in \mathscr{W}^{\mathfrak{s}}_{1,\star}$ and an open neighborhood U of $\mathbf{p}_\star \in \mathscr{P}$, there exists a loop $(\mathbf{p}_t)_{t \in S^1}$ in $U \cap (\mathscr{P} \backslash \mathscr{W}^{\mathfrak{s}}_b)$ such that:*

1. *$\mathbf{p}_{1/4}, \mathbf{p}_{3/4} \in \mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$, and $\mathbf{p}_t \in \mathscr{P} \backslash \mathscr{W}^{\mathfrak{s}}_1$ for all $t \notin \{1/4, 3/4\}$,*
2. *if $\Psi_{1/4}$ and $\Psi_{3/4}$ denote spinors spanning $\not{D}^{\mathfrak{s}}_{\mathbf{p}_{1/4}}$ and $\not{D}^{\mathfrak{s}}_{\mathbf{p}_{3/4}}$, then*
$$\deg(\Psi_{1/4}, \Psi_{3/4}) = \pm 1;$$
and
3. *the spectral crossings at $\mathbf{p}_{1/4}$ and $\mathbf{p}_{3/4}$ occur with opposite signs.*

In particular, the intersection number of $(\mathbf{p}_t)_{t \in [0,1]}$ with $\mathscr{W}^{\mathfrak{s}}_{1,\varnothing}$ with respect to the coorientation from Proposition 3.10.2 is ± 2.

Proof. Let $\Psi_\star \in \ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_\star}$ and $x_\star \in M$ be such that $\|\Psi_\star\|_{L^2} = 1$ and $\Psi_\star(x_\star) = 0$. Let $\phi \in \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E))$ be such that

$$\begin{aligned} &\mathrm{im}((\nabla \Psi_\star)_{x_\star}) + \mathbf{R}\langle \phi_\star(x_\star) \rangle = \mathrm{Re}(S_{\mathfrak{s}} \otimes E)_{x_\star}, \\ &|\phi_\star(x_\star)| = 1, \quad \text{and} \quad \nabla \phi(x_\star) = 0. \end{aligned}$$

We can assume that U is sufficiently small for the Implicit Function Theorem to guarantee that there is a unique smooth map $U \to \mathbf{R} \times \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E)) \times M \times \mathbf{R}$,

$$\mathbf{p} \mapsto (\lambda(\mathbf{p}), \Psi_{\mathbf{p}}, x_{\mathbf{p}}, \nu(\mathbf{p}))$$

such that

$$\lambda(\mathbf{p}_\star) = 0, \quad \Psi_{\mathbf{p}_\star} = \Psi_\star, \quad \text{and} \quad x_{\mathbf{p}_\star} = x_\star$$

as well as

$$\not{D}^{\mathfrak{s}}_{\mathbf{p}} \Psi_{\mathbf{p}} = \lambda(\mathbf{p}) \Psi_{\mathbf{p}} \quad \Psi_{\mathbf{p}}(x_{\mathbf{p}}) = \nu(\mathbf{p}) \phi_\star(x_{\mathbf{p}}), \quad \text{and} \quad \|\Psi_0\|_{L^2} = 1. \tag{3.10.3}$$

As before

$$U \cap \mathscr{W}_1^{\mathfrak{s}} = \lambda^{-1}(0).$$

Set

$$\mathscr{N} := \nu^{-1}(0).$$

This is the set of those $\mathbf{p} \in U$ for which the eigenspinor with smallest eigenvalue has a unique zero which is also non-degenerate.

From the proof of Proposition 3.10.2 we know that $U \cap \mathscr{W}_1^{\mathfrak{s}}$ is a codimension one submanifold. We will now show that $\mathscr{N}$ is a codimension one submanifold as well and that it intersects $U \cap \mathscr{W}_1^{\mathfrak{s}}$ transversely in $U \cap \mathscr{W}_{1,\star}^{\mathfrak{s}}$, see Figure 3.2. Knowing this, the existence of a loop $(\mathbf{p})_{t \in S^1}$ with the desired properties follows easily because crossing $\mathscr{N}$ changes the relative degree by ± 1. Indeed, let t_0 be a time at which the path crosses $\mathscr{N}$ and let ε be a small positive number. Consider the path of eigenspinors $(\Psi_{\mathbf{p}_t})$ for $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$ as introduced in (3.10.3). For $t \neq t_0$, each of the spinors is nowhere vanishing and $\Psi_{\mathbf{p}_{t_0}}$ has a single non-degenerate zero. Thus, $\deg(\Psi_{\mathbf{p}_{t_0-\varepsilon}}, \Psi_{\mathbf{p}_{t_0+\varepsilon}}) = \pm 1$.

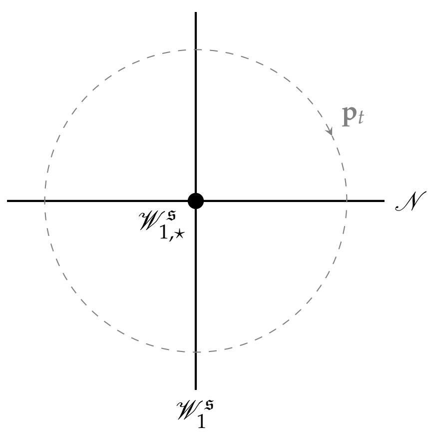


Figure 3.2: $\mathscr{W}_1^{\mathfrak{s}}$ and $\mathscr{N}$ intersecting in $\mathscr{W}_{1,\star}^{\mathfrak{s}}$.

We will show that $\mathrm{d}_{\mathbf{p}_\star}\nu|_{T_{\mathbf{p}_\star}\mathscr{W}_1^{\mathfrak{s}}}$ is non-vanishing. This implies both that $\mathscr{N}$ is a codimension one manifold and that it intersects $U \cap \mathscr{W}^{\mathfrak{s}_1}$ transversely. For $\hat{\mathbf{p}} = (0, b) \in T_{\mathbf{p}_\star}\mathscr{P}$ to be determined, set

$$\lambda_t := \lambda(\mathbf{p}_\star + t\hat{\mathbf{p}}), \quad \Psi_t := \Psi_{\mathbf{p}_\star + t\hat{\mathbf{p}}}, \quad x_t := x_{\mathbf{p}\star + t\hat{\mathbf{p}}}, \quad \text{and} \quad \nu_t := \nu_{\mathbf{p}_\star + t\hat{\mathbf{p}}}$$

as well as

$$\dot\lambda := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\lambda_t, \quad \dot\Psi := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Psi_t, \quad \dot x := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}x_t, \quad \text{and} \quad \dot\nu := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\nu_t.$$

Differentiating (3.10.3) we obtain

$$\bar\gamma(b)\Psi_\star + \not{D}^{\mathfrak{s}}_{\mathbf{p}_\star}\dot\Psi = \dot\lambda\Psi_\star, \quad \dot\Psi(x_\star) + (\nabla\Psi_\star)_{x_\star}\dot x = \dot\nu\phi(x_\star), \quad \text{and} \quad \langle\Psi_\star, \dot\Psi\rangle_{L^2} = 0.$$

From this it follows that

$$\begin{aligned}\dot\lambda &= \langle\bar\gamma(b)\Psi_\star, \Psi_\star\rangle_{L^2}, \\ \not{D}^{\mathfrak{s}}_{\mathbf{p}_\star}\dot\Psi &= \bar\gamma(b)\Psi_\star - \langle\bar\gamma(b)\Psi_\star, \Psi_\star\rangle_{L^2}\Psi_\star, \quad \text{and} \\ \dot\nu &= \langle\dot\Psi(x_\star), \phi(x_\star)\rangle.\end{aligned} \tag{3.10.4}$$

Suppose we can arrange a choice of b such that

1. $\langle\bar\gamma(b)\Psi_\star, \Psi_\star\rangle_{L^2} = 0$,
2. $\not{D}^{\mathfrak{s}}_{\mathbf{p}_\star}\dot\Psi$ vanishes in a neighborhood of $x_\star$.
3. $\langle\dot\Psi(x_\star), \phi(x_\star)\rangle \neq 0$, and
4. $\langle\Psi_\star, \dot\Psi\rangle_{L^2} = 0$.

In this situation it would follow that

$$\dot\lambda = 0 \quad \text{but} \quad \dot\nu \neq 0;$$

that is

$$\hat{\mathbf{p}} = (0, b) \in T_{\mathbf{p}_\star}\mathscr{W}^{\mathfrak{s}}_1 \quad \text{and} \quad \mathrm{d}_{\mathbf{p}_\star}\nu(b) \neq 0.$$

It remains to find such a b. To begin with, observe that we can certainly find $\dot\Psi$ with the above properties by solving the Dirac equation in a neighborhood of $x_\star$ subject to the constraint (3) and then extending to all of M so that (4) holds. Fix such a choice of $\dot\Psi$. Clifford multiplication by $T^*M \otimes \mathfrak{su}(E)$ on $\mathrm{Re}(S_{\mathfrak{s}} \otimes E)$ induces a isomorphism between $T^*M \otimes \mathfrak{su}(E)$ and trace-free symmetric endomorphisms of $\mathrm{Re}(S_{\mathfrak{s}} \otimes E)$. Since $\not{D}^{\mathfrak{s}}_{\mathbf{p}_\star}\dot\Psi$ vanishes in a neighborhood of $x_\star$ and $\Psi_\star$ vanishes only at $x_\star$, one can find $b \in \Omega^1(M, \mathfrak{su}(E))$ such that

$$\langle\bar\gamma(b)\Psi_\star, \Psi_\star\rangle_{L^2} = 0 \quad \text{and} \quad \bar\gamma(b)\Psi_\star = \not{D}^{\mathfrak{s}}_{\mathbf{p}_\star}\dot\Psi.$$

This completes the proof. □

It remains to exhibit a point $\mathbf{p}_\star \in \mathscr{W}^{\mathfrak{s}}_{1,\star}$ for some spin structure $\mathfrak{s}$ but such that $\mathbf{p}_\star \notin \mathscr{W}^{\tilde{\mathfrak{s}}}$ for every other spin structure $\tilde{\mathfrak{s}}$. This requires the following two propositions as preparation.

Proposition 3.10.6. *Let $k \in \{2, 3, \ldots\}$. The subset*

$$\mathscr{W}^{\mathfrak{s}}_k := \{\mathbf{p} \in \mathscr{P} : \dim\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} = k\} \subset \mathscr{P}$$

is contained in a submanifold of codimension three. Moreover, $\mathscr{W}^{\mathfrak{s}}_k \cap \overline{\mathscr{W}^{\mathfrak{s}}_1} = \mathscr{W}^{\mathfrak{s}}_k$.

Proof. Let $\mathbf{p}_0 \in \mathscr{P}$ such that $\dim\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_0} = k$. Choose an L^2–orthonormal basis $\{\Psi_i\}$ of $\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}_0}$. For a sufficiently small neighborhood U of $\mathbf{p}_0$, by the Implicit Function Theorem, there exists a unique smooth map $U \to \Gamma(\mathrm{Re}(S_{\mathfrak{s}} \otimes E))^{\oplus^k} \times S^2\mathbf{R}^k$

$$\mathbf{p} \mapsto \big(\Psi_{1,\mathbf{p}}, \dots, \Psi_{k,\mathbf{p}}, \Lambda(\mathbf{p}) = (\lambda_{ij}(\mathbf{p}))\big)$$

such that

$$\Psi_{i,\mathbf{p}_0} = \Psi_i \quad \text{and} \quad \Lambda(\mathbf{p}_0) = 0$$

as well as

$$\not{D}^{\mathfrak{s}}_{\mathbf{p}}\Psi_{i,\mathbf{p}} = \sum_{j=1}^{k} \lambda_{ij}\Psi_{j,\mathbf{p}} \quad \text{and} \quad \langle \Psi_{i,\mathbf{p}}, \Psi_{j,\mathbf{p}} \rangle_{L^2} = \delta_{ij}.$$

(It follows from the fact that $\not{D}^{\mathfrak{s}}_{\mathbf{p}}$ is symmetric, that $\lambda_{ij} = \lambda_{ji}$.) We have

$$U \cap \mathscr{W}^{\mathfrak{s}}_k = \Lambda^{-1}(0)$$

We will show that $\mathrm{d}\Lambda\colon T_{\mathbf{p}_0}U \to S^2\mathbf{R}^k$ has rank at least three. This will imply that $\mathscr{W}^{\mathfrak{s}}_k$ has codimension at least three.

Suppose $\Psi_2 = f\Psi_1$ for some function $f \in C^\infty(M)$. It follows that

$$0 = \not{D}^{\mathfrak{s}}_{\mathbf{p}_0}\Psi_2 = \gamma(\nabla f)\Psi_1$$

This in turn implies that f is constant because Ψ_1 is non-vanishing on an dense open subset of M. However, this is non-sense because $\langle \Psi_i, \Psi_j \rangle_{L^2} = \delta_{ij}$. It follows that there is an $x \in M$ such that $\Psi_1(x)$ and $\Psi_2(x)$ are linearly independent. Clifford multiplication induces an isomorphism from $T^*M \otimes \mathfrak{su}(E)$ to trace-free symmetric endomorphisms of $\mathrm{Re}(S_{\mathfrak{s}} \otimes E)$. Therefore, given any $(\mu_{ij}) \in S^2\mathbf{R}^2$, we can find $\hat{\mathbf{p}} = (0, b) \in T_{\mathbf{p}_0}U$ such that

$$\langle \bar{\gamma}(b)\Psi_i, \Psi_j \rangle_{L^2} = \mu_{ij} \quad \text{for} \quad i, j \in \{1, 2\}.$$

Since

$$\mathrm{d}_{\mathbf{p}_0}\Lambda(\hat{\mathbf{p}}) = \big(\langle \bar{\gamma}(b)\Psi_i, \Psi_j \rangle_{L^2}\big) \in S^2\mathbf{R}^k,$$

it follows that $\mathrm{d}_{\mathbf{p}_0}\Lambda$ has rank at least three.

It follows from the above that, for any $\mathbf{p}_0 \in \mathscr{W}^{\mathfrak{s}}_k$, there exists an arbitarily close $\mathbf{p} \in \mathscr{P}$ with $0 < \dim\ker \not{D}^{\mathfrak{s}}_{\mathbf{p}} < k$. From this it follows by induction that $\mathscr{W}^{\mathfrak{s}}_k \cap \overline{\mathscr{W}^{\mathfrak{s}}_1} = \mathscr{W}^{\mathfrak{s}}_k$. □

Proposition 3.10.7. *If $\mathfrak{s}_1, \mathfrak{s}_2$ are two distinct spin structures, then $\mathscr{W}^{\mathfrak{s}_1}_1$ and $\mathscr{W}^{\mathfrak{s}_2}_1$ intersect transversely.*

Proof. Let $\mathbf{p} \in \mathscr{W}^{\mathfrak{s}_1}_1 \cap \mathscr{W}^{\mathfrak{s}_2}_1$. Denote by Ψ_1 and Ψ_2 spinors spanning $\ker \not{D}^{\mathfrak{s}_1}_{\mathbf{p}}$ and $\ker \not{D}^{\mathfrak{s}_2}_{\mathbf{p}}$ respectively. The spin structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ differ by twisting by a $\mathbf{Z}_2$–bundle $\mathfrak{l}$. This bundle corresponds to a double cover $\pi\colon \tilde{M} \to M$ and upon pulling back to the cover the spin structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ both correspond to the same spin structure $\tilde{\mathfrak{s}}$. Let $\tilde{\Psi}_i = \pi^*\Psi_i$ for $i = 1, 2$ be the lifts of Ψ_1, Ψ_2 to $\tilde{M}$. The natural involution σ on $S_{\tilde{\mathfrak{s}}} \to \tilde{M}$ acts as -1 on $\tilde{\Psi}_1$ and as $+1$ on $\tilde{\Psi}_2$. In particular, we have

$$\langle \tilde{\Psi}_1, \tilde{\Psi}_2 \rangle_{L^2} = \langle \sigma(\tilde{\Psi}_1), \sigma(\tilde{\Psi}_2) \rangle_{L^2} = \langle -\tilde{\Psi}_1, \tilde{\Psi}_2 \rangle_{L^2} = -\langle \tilde{\Psi}_1, \tilde{\Psi}_2 \rangle_{L^2};$$

hence, $\langle \tilde{\Psi}_1, \tilde{\Psi}_2 \rangle_{L^2} = 0$. After renormalization, we can assume that $\langle \tilde{\Psi}_i, \tilde{\Psi}_j \rangle_{L^2} = \delta_{ij}$. It follows from the argument used in the proof of Proposition 3.10.6 that there is an $x \in M$ such that, for $\tilde{x} \in \pi^{-1}(x)$, $\tilde{\Psi}_1(\tilde{x})$ and $\tilde{\Psi}_2(\tilde{x})$ are linearly independent.

Let $\lambda^{\mathfrak{s}_1}$ and $\lambda^{\mathfrak{s}_2}$ be the local defining functions for the walls $\mathscr{W}_1^{\mathfrak{s}_1}$ and $\mathscr{W}_1^{\mathfrak{s}_2}$ respectively, defined via (3.10.1) in the proof of Proposition 3.10.2. The derivative of $\lambda^{\mathfrak{s}_i}$ in the direction of $b \in \Omega^1(M, \mathfrak{su}(E))$ is given by (3.10.2):

$$\mathrm{d}_{\mathbf{p}} \lambda^{\mathfrak{s}_i}(0, b) = \langle \bar{\gamma}(b) \Psi_i, \Psi_i \rangle_{L^2} = \frac{1}{2} \langle \bar{\gamma}(\pi^* b) \tilde{\Psi}_i, \tilde{\Psi}_i \rangle_{L^2}.$$

Since $\tilde{\Psi}_1(\tilde{x})$ and $\tilde{\Psi}_2(\tilde{x})$ are linearly independent, there exists $b(x) \in T_x M \otimes \mathfrak{su}(E_x)$ such that

$$\bar{\gamma}(\pi^* b(x)) \tilde{\Psi}_1(\tilde{x}) = 0 \quad \text{and} \quad \bar{\gamma}(\pi^* b(x))) \tilde{\Psi}_2(\tilde{x}) = \tilde{\Psi}_2(\tilde{x})$$

We extend $b(x)$ to a section $b \in \Omega^1(M, \mathfrak{su}(E))$ such that

$$\mathrm{d}_{\mathbf{p}} \lambda^{\mathfrak{s}_1}(0, b) = 0 \quad \text{but} \quad \mathrm{d}_{\mathbf{p}} \lambda^{\mathfrak{s}_2}(0, b) \neq 0.$$

This shows that derivatives of the local defining functions of $\mathscr{W}_1^{\mathfrak{s}_1}$ and $\mathscr{W}_1^{\mathfrak{s}_2}$ are linearly independent; hence, the walls intersect transversely. □

Finally, we are in a position to construct $\mathbf{p}_\star \in \mathscr{W}_{1,\star}^{\mathfrak{s}}$. Fix a spin structure $\mathfrak{s}$ as well as $\mathbf{p}_0 = (g_0, B_0)$ and $x_\star \in M$ such that g_0 and B_0 are flat on a small ball around a point $x_\star \in M$. Choose local coordinates (y_1, y_2, y_3) around $x_\star$ and a local trivialization of $\mathrm{Re}(S_{\mathfrak{s}} \otimes E)$ in which g_0 is given by the identity matrix and B_0 is the trivial connection. Let $\Psi \in \Gamma(\mathrm{Re} \otimes S_{\mathfrak{s}})$ be any section which is nowhere vanishing away from $x_\star$ and around $x_\star$ agrees with the map $\mathbf{R}^3 \to \mathbf{H}$ given by

$$(y_1, y_2, y_3) \mapsto 2iy_1 - jy_2 - ky_3.$$

In particular, Ψ has a single non-degenerate zero at $x_\star$ and satisfies $\bar{D}_{\mathbf{p}}^{\mathfrak{s}} \Psi = 0$ in a neighborhood of $x_\star$. Using the same argument as in the proof of Proposition 3.10.5, we find $b \in \Omega^1(\mathfrak{su}(E))$ vanishing in a neighborhood of $x_\star$ and such that for $\mathbf{p}_\star = (g_0, B_0 + b)$ we have

$$0 = \bar{D}_{\mathbf{p}_\star}^{\mathfrak{s}} \Psi = \bar{D}_{\mathbf{p}_0}^{\mathfrak{s}} \Psi + \bar{\gamma}(b) \Psi.$$

This shows that Ψ is harmonic with respect to $\mathbf{p}_\star$. If $\dim \ker \bar{D}_{\mathbf{p}_\star} > 1$, then Proposition 3.10.6 and the argument from Proposition 3.10.5 can be used to slightly perturb $\mathbf{p}_\star$ to arrange that $\dim \ker \bar{D}_{\mathbf{p}_\star} = 1$ and any spinor spanning $\bar{D}_{\mathbf{p}_\star}$ has a non-degenerate zero (close to $x_\star$). Similarly, Proposition 3.10.6 and Proposition 3.10.7 can be used to ensure that there are no non-trivial harmonic spinors with respect to $\mathbf{p}_\star$ for any other spin structure $\tilde{\mathfrak{s}}$. □

3.11 RELATION TO GAUGE THEORY ON G2–MANIFOLDS

Donaldson and Thomas [DT98, Section 3] suggested that one might be able to construct G_2 analogues of the Casson invariant/instanton Floer homology associated to a natural functional whose critical point are G_2–instantons. One key difficulty with this proposal is that G_2–instantons can degenerate by bubbling along associative submanifolds. Donaldson and Segal [DS11, Section 6] explain that this bubbling could be caused by the appearance of (nowhere vanishing) harmonic spinors of $\mathrm{Re}(E \otimes S)$ over associative submanifolds. In particular, the signed count of G_2–instantons can jump along a one-parameter family. Donaldson and Segal propose to compensate this jump with a counter-term consisting of a weighted count of associative submanifolds.

Joyce [Joy17, Example 8.5] poses the following scenario. Consider a one–parameter family of G_2–manifolds $\{(Y, \phi_t) : t \in [0,1]\}$ together with an SU(2)–bundle E where:

- there is a smooth family of irreducible connections $(A_t)_{t\in[0,1]} \in \mathscr{A}(E)^{[0,1]}$ such that A_t is an unobstructed G_2–instanton with respect to ϕ_t for each $t \in [0,1]$,
- there are no relevant associatives in (Y, ϕ_t) for $t \in [0, 1/3) \cup (2/3, 1]$, and
- there is an obstructed associative $P_{1/3}$ in $(Y, \phi_{1/3})$, which splits into two unobstructed associatives $P_t^\pm$ in (Y, ϕ_t) for $t \in (1/3, 2/3)$, and which then annihilate each other in an obstructed associative $P_{2/3}$ in $(Y, \phi_{2/3})$.

According to [Wal17, Theorem 1.2] a regular crossing of the spectral flow of family of Dirac operators $\not{D}_t^\pm \colon \Gamma(\mathrm{Re}(E|_{P_t^\pm} \otimes \not{S}_{P_t^\pm})) \to \Gamma(\mathrm{Re}(E|_{P_t^\pm} \otimes \not{S}_{P_t^\pm}))$ causes a jump in the signed count of G_2–instantons; however, the sign of this jump has not been analyzed.[7] Donaldson and Segal [DS11, Section 6] and Joyce [Joy17, Section 8.4] suggest that this is the only source of jumping phenomena. The difference in the spectral flows of the Dirac operators $\not{D}_t^\pm$ is a topological invariant, say $k \in \mathbf{Z}$, which may be non-zero. [Joy17, Section 8.4] thus concludes that passing from $t < 1/3$ to $t > 2/3$ the signed number of G_2–instantons should change by $k \cdot |H_1(P_{1/3}, \mathbf{Z}_2)|$; and, since there are no associatives for $t \in [0, 1/3) \cup (2/3, 1]$, no counter-term involving a weighted count of associatives could compensate this jump.

It is proposed in [HW15] that the weight associated with each associative 3-manifold should be the signed count of solutions to the Seiberg–Witten equation with two spinors. The loop of associatives can equivalently be seen as a path of parameters $(\mathbf{p}_t)_{t\in[0,1]}$ on a fixed 3–manifold P, with $\mathbf{p}_1$ gauge equivalent to $\mathbf{p}_0$. Therefore, one can ask how $n(\mathbf{p}_t)$ varies in this scenario. Suppose that $b_1(P_{1/3}) > 1$. Assuming there are no harmonic $\mathbf{Z}_2$ spinors along the path $(\mathbf{p}_t)_{t\in[0,1]}$, a jump in $n(\mathbf{p}_t)$ would occur precisely when the spectrum of one of the Dirac operators $\not{D}_{\mathbf{p}_t}^{\mathfrak{s}}$ crosses zero. If the wall-crossing formula for $n(\mathbf{p}_t)$ were given by the sum of the spectral flows of $(\not{D}_{\mathbf{p}_t}^{\mathfrak{s}})_{t\in[0,1]}$, then we would have arrive a contradiction just like in Joyce's argument:

$$0 \neq k \cdot |H_1(P, \mathbf{Z}_2)| = n(\mathbf{p}_1) - n(\mathbf{p}_0) = 0$$

since $\mathbf{p}_1$ and $\mathbf{p}_0$ are gauge equivalent. However, the conclusion of our work is that:

7 To be more precise, the jump occurs in the signed count of G_2–instantons on a bundle E', which is related to E by $c_2(E') = c_2(E) + \mathrm{PD}[P]$ with $[P] = [P_{1/3}] = [P_t^\pm] = [P_{2/3}]$.

1. The wall-crossing for $n(\mathbf{p}_t)$ caused by harmonic spinors is not given by the spectral flow.
2. There exist singular harmonic $\mathbf{Z}_2$ spinors which cause additional wall-crossing.

It is possible that the same happens for the signed count of G_2–instantons. To evaluate the viability of the proposal in [HW15] it is important to answer the following questions.

Question 3.11.1. What is the sign of the jump in the number of G_2–instantons caused by a harmonic spinor?

Question 3.11.2. Do singular harmonic $\mathbf{Z}_2$ spinors cause a jump in the number of (possibly singular) G_2–instantons?

4 ADHM MONOPOLES ON RIEMANN SURFACES

We have seen that for a general quaternionic representation ρ, the moduli space of ρ–monopoles can be non-compact. In Chapter 2 we constructed its partial compactification consisting of ρ–monopoles and Fueter sections with values in the hyperkähler quotient associated with ρ. However, the existence result of Chapter 3 shows that the full compactification should include also Fueter sections which are singular along 1–dimensional sets. In particular, one expects that there is an analogue of the wall-crossing phenomenon, described in the previous chapters, which involves such singular Fueter sections. A precise construction of the full compactification of the moduli space of ρ–monopoles is a central open problem in the study of the Seiberg–Witten equations associated with quaternionic representations.

The purpose of Chapter 4 is to focus on examples, which give us new insights into the general problem of constructing such a compactification. Specifically, we study ADHM monopoles on Riemannian 3–manifolds of the form $M = S^1 \times \Sigma$ for a compact Riemann surface Σ.

In Section 4.1 we prove that for any quaternionic representation ρ, all irreducible ρ–monopoles on $S^1 \times \Sigma$ are pulled back from solutions of a vortex-type equation on Σ. In Section 4.2 we apply this theorem to the ADHM Seiberg–Witten equations. This establishes a Hitchin–Kobayashi correspondence between ADHM monopoles on $S^1 \times \Sigma$ and certain holomorphic objects on Σ satisfying a stability condition. It turns out that such holomorphic objects on Riemann surfaces have been already studied in algebraic geometry under the name of *ADHM bundles*.

In Section 4.3 we specialize further to the Seiberg–Witten equation with two spinors. In this setting we construct an isomorphism, in the category of real analytic spaces, between the moduli space $\mathfrak{M}$ of monopoles with two spinors and the moduli space $\mathfrak{M}^{\mathrm{hol}}$ of the holomorphic data appearing in the Hitchin–Kobayashi correspondence. Section 4.4 is devoted to the problem of compactifying these moduli spaces. We construct two compactifications $\mathfrak{M} \subset \overline{\mathfrak{M}}$ and $\mathfrak{M}^{\mathrm{hol}} \subset \overline{\mathfrak{M}}^{\mathrm{hol}}$, using gauge theory and complex geometry respectively, and prove that the isomorphism $\mathfrak{M} \cong \mathfrak{M}^{\mathrm{hol}}$ extends to a homeomorphism $\overline{\mathfrak{M}} \cong \overline{\mathfrak{M}}^{\mathrm{hol}}$.[1] A consequence of this result is that $\overline{\mathfrak{M}}$ is a compact analytic spaces containing the original moduli space $\mathfrak{M}$ as a Zariski open, dense subset.

In Section 4.5 and Section 4.6 we consider the problem counting monopoles with two spinors on $M = S^1 \times \Sigma$. This involves studying the existence of harmonic $\mathbf{Z}_2$ spinors. We show that for a generic choice of S^1–invariant parameters of the Seiberg–Witten equation with two spinors, harmonic $\mathbf{Z}_2$ spinors do not appear and $\mathfrak{M} = \overline{\mathfrak{M}}$. Moreover, in this case $\mathfrak{M}$ is a compact Kähler manifold. This manifold is often of positive dimension, because monopoles in $\mathfrak{M}$ are obstructed in the sense of deformation theory from Section 2.3 After a small perturbation,

1 In Chapter 2, we used the symbol $\overline{\mathfrak{M}}$ to denote the partial compactification. Throughout this chapter we use the same symbol for a larger space containing also certain singular harmonic $\mathbf{Z}_2$ spinors. This should not cause any confusion as the partial compactification from Chapter 2 never appears in this chapter.

which is no longer S^1–invariant, all monopoles in $\mathfrak{M}$ become unobstructed and the moduli space splits into isolated points. These points can be counted with signs, yielding a number independent of the initial choice of S^1–invariant parameters of the Seiberg–Witten equation. In Section 4.7 we compute the resulting signed count of monopoles with two spinors when Σ has genus zero, one, and two. In addition, we construct many explicit examples of monopoles with two spinors and harmonic $\mathbf{Z}_2$ spinors using complex geometry.

From the viewpoint of higher-dimensional gauge theory, the case $M = S^1 \times \Sigma$ considered in this chapter is relevant to the study of G_2–manifolds of the form $X = S^1 \times Z$. Here Z is a Calabi–Yau 3–fold and Σ is embedded in Z as a holomorphic curve. G_2–instantons over $S^1 \times Z$ correspond to Hermitian–Yang–Mills connections on Z and so one expects that there is a relationship between monopoles on $S^1 \times \Sigma$ and Hermitian–Yang–Mills connections on Z whose energy is highly concentrated around Σ. By the Donaldson–Uhlenbeck–Yau theorem, Hermitian–Yang–Mills connections correspond to stable holomorphic vector bundles. Thus, the results of this chapter can be seen as the first step towards a gauge-theoretic interpretation of local Donaldson–Thomas invariants in algebraic geometry [OP10; Dia12b]; see also [DW17b, Section 7] for more details on the relationship between ADHM monopoles and Donaldson–Thomas theory.

REFERENCES This chapter is a rewritten version of the article [Doa17b], incorporating also some material from [Doa17a]. Section 4.2 is based on [DW17b, Section 7.4], written in collaboration with Thomas Walpuski.

4.1 A DIMENSIONAL REDUCTION

Let Σ be a compact, connected surface and let $M = S^1 \times \Sigma$. In this section, we study Seiberg–Witten equations on M under the assumption that the parameters of the equations are invariant in the S^1–direction. The main result of this section is Theorem 4.1.12 below. The classical Seiberg–Witten equation was studied in the S^1–invariant setting by Morgan, Szabó, and Taubes [MST96], Mrowka, Ozsváth, and Yu [MOY97], and Muñoz and Wang [MW05].

In what follows we will use the notation and definitions introduced in Section 2.2. Suppose that a set of algebraic data as in Definition 2.1.7 was chosen. Equip M with a set of compatible geometric data as in Definition 2.2.1, which is pulled back from Σ in the following sense.

Definition 4.1.1. A set of geometric data $(g, \mathfrak{s}, Q, B)$ as in Definition 2.2.1 is *pulled back from* Σ if

1. the Riemannian metric g is the product of the standard metric on S^1 and a Riemannian metric g_Σ on Σ,
2. the spin structure $\mathfrak{s}$ is induced from a spin structure on Σ,
3. the principal H–bundle $Q \to M$ is pulled back from a bundle on Σ,
4. the connection B is pulled back from a connection on Σ.

Remark 4.1.2. We could work under the weaker assumption that only the principal K–bundle $R \to M$ is pulled back from Σ. In Theorem 4.1.12 below we prove that this assumption and the existence of an irreducible solution of the Seiberg–Witten equation implies that $Q \to M$ must be pulled back from Σ.

Recall that a choice of a spin structure on Σ is equivalent to a choice of a Hermitian line bundle $K^{1/2}$ together with a $\mathrm{U}(1)$ connection and an isomorphism $(K^{1/2})^{\otimes 2}$ with $K = \Lambda^{1,0}T^*\Sigma$, the canonical bundle of the Riemann surface (Σ, g). The corresponding spin structure $\mathfrak{s}$ on $M = S^1 \times \Sigma$ is isomorphic, as a principal $\mathrm{Sp}(1)$–bundle, to the pull back of the $\mathrm{Sp}(1)$–bundle on Σ associated with $K^{-1/2}$ via the standard inclusion $\mathrm{U}(1) \subset \mathrm{Sp}(1)$. The spinor bundle of $\mathfrak{s}$ is the pull back of the rank two complex vector bundle $K^{1/2} \oplus K^{-1/2}$.

Remark 4.1.3. To keep the notation simple, when it is not likely to cause confusion, we will use the same symbols $K^{1/2}$, Q, B, and so on, for the corresponding bundles and connections on Σ and their pull backs to M.

In the situation at hand, the generalized spinor bundle $\mathfrak{S}$ over M, introduced in Definition 2.2.3, is pulled back from the following bundle over Σ:

$$\mathfrak{S} = (K^{-1/2} \times Q) \times_{\mathrm{U}(1) \times H} S.$$

The action of unit quaternions $\mathrm{Sp}(1)$ on the quaternionic vector space S rotates the sphere of complex structures, with $\mathrm{U}(1) \subset \mathrm{Sp}(1)$ being the stabiliser of i; thus, $\mathfrak{S}$ is a complex vector bundle. Consider the quaternionic vector bundle $V = Q \times_H S$; then

$$\mathfrak{S} = V \otimes_{\mathbf{C}} K^{-1/2},$$

where the complex structure on V is given by i. The remaining part of the quaternionic structure is encoded in an anti-linear involution $j \colon V \to V$. Taking the tensor product of j with the anti-linear map $K^{1/2} \to K^{-1/2}$ given by the metric, we obtain an anti-linear isomorphism

$$\sigma \colon V \otimes K^{1/2} \to V \otimes K^{-1/2}.$$

We define similarly a map going in the opposite direction, also denoted by σ, so that $\sigma^2 = -1$. Equivalently, σ can be seen as a map

$$\sigma \colon \mathfrak{S} \otimes K \to \mathfrak{S}$$

which is a two-dimensional manifestation of the Clifford multiplication.

Next, we relate sections and connections on M to those on Σ.

Definition 4.1.4. Denote by $\mathscr{A}_B(M, Q)$ the space of connections on $Q \to M$ inducing the connection B on R, as in Definition 2.2.4. Denote by $\mathscr{A}_B(\Sigma, Q)$ the analogous space of connections on $Q \to \Sigma$. Using the pull back operation, we interpret $\mathscr{A}_B(\Sigma, Q)$ as a subset of $\mathscr{A}_B(M, Q)$.

Let $t \in [0, 1]$ be the coordinate on S^1 in $M = S^1 \times \Sigma$. Any connection $A_M \in \mathscr{A}_B(M, Q)$ can be uniquely written in the form

$$A_M = A(t) + b(t)\mathrm{d}t$$

for one-periodic families $A(t)$ of connections in $\mathscr{A}_B(\Sigma, Q)$ and sections $b(t)$ of the adjoint bundle $\mathfrak{g}_P$ introduced in Definition 2.2.4. The subset $\mathscr{A}_B(\Sigma, Q) \subset \mathscr{A}_B(M, Q)$ consists of those A_M for which $b(t) = 0$.

Likewise, any section $\Phi \in \Gamma(M, \mathfrak{S})$ can be identified with a one-periodic family $\Phi(t) \in \Gamma(\Sigma, \mathfrak{S})$. Thus, $\Gamma(\Sigma, \mathfrak{S})$ embeds into $\Gamma(M, \mathfrak{S})$ as sections independent of t. Finally, the group of restricted gauge transformations $\mathscr{G}(\Sigma, P)$, defined as in Definition 2.2.12, is naturally a subgroup of $\mathscr{G}(M, P)$.

Definition 4.1.5. An S^1–*invariant configuration* is an element of

$$\Gamma(\Sigma, \mathfrak{S}) \times \mathscr{A}_B(\Sigma, Q) \subset \Gamma(M, \mathfrak{S}) \times \mathscr{A}_B(M, Q).$$

Proposition 4.1.6. *If two S^1–invariant configurations differ by $g \in \mathscr{G}(M, P)$, then $g \in \mathscr{G}(\Sigma, P)$. In particular,*

$$\frac{\Gamma(\Sigma, \mathfrak{S}) \times \mathscr{A}_B(\Sigma, Q)}{\mathscr{G}(\Sigma, P)}$$

is a submanifold of

$$\frac{\Gamma(M, \mathfrak{S}) \times \mathscr{A}_B(M, Q)}{\mathscr{G}(M, P)}.$$

The Dirac operator on M can be expressed in terms of the Dolbeault operator on Σ. Consider the simplest case $M = \mathbf{R}^3 = \mathbf{R} \times \mathbf{C}$ and $S = \mathbb{H}$. Denoting coordinates on $\mathbf{R} \times \mathbf{C}$ by t and $z = x + iy$, we have for a map $\Phi \colon \mathbb{R}^3 \to \mathbb{H}$

$$\not{D}\Phi = i\frac{\partial \Phi}{\partial t} + j\frac{\partial \Phi}{\partial x} + k\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial t} + j\left(\frac{\partial \Phi}{\partial x} - i\frac{\partial \Phi}{\partial y}\right) = i\frac{\partial \Phi}{\partial t} + 2j\frac{\partial \Phi}{\partial z}. \quad (4.1.1)$$

In general, $\partial/\partial z$ is replaced by the Dolbeault operator

$$\partial_A \colon \Gamma(\Sigma, \mathfrak{S}) \to \Gamma(\Sigma, \mathfrak{S} \otimes K)$$

or equivalently,

$$\partial_A \colon \Gamma(\Sigma, V \otimes K^{-1/2}) \to \Gamma(\Sigma, V \otimes K^{1/2}),$$

which is defined as the $(1,0)$–part of the covariant derivative ∇_A acting on sections of $\mathfrak{S}$. The proof of the next proposition is a simple calculation in conformal coordinates, almost the same as (4.1.1). [2]

Proposition 4.1.7. *Let $A_M = A(t) + b(t)\mathrm{d}t$ be a connection in $\mathscr{A}(M, P)$ and $\Phi = \Phi(t)$ a section in $\Gamma(M, \mathfrak{S})$. The Dirac operator $\not{D}_{A_M}$ acting on $\Gamma(M, \mathfrak{S})$ is given by*

$$\not{D}_{A_M}\Phi = i\left(\frac{\partial \Phi}{\partial t} + b(t)\Phi\right) + \sqrt{2}\sigma\left(\partial_{A(t)}\Phi\right).$$

We move on to the hyperkähler moment map $\mu \colon S \to \mathfrak{g} \otimes \mathbf{R}^3$ introduced in Section 2.1. In the dimensionally-reduced setting, we use the splitting of μ into the real and complex parts: $\mu_{\mathbf{R}} \colon S \to \mathfrak{g}$ and $\mu_{\mathbf{C}} \colon S \to \mathfrak{g} \otimes \mathbf{C}$. If $\mu = (\mu_i, \mu_j, \mu_k)$ are the three components of μ, then $\mu_{\mathbf{R}} = \mu_i$ and $\mu_{\mathbf{C}} = \mu_j + i\mu_k$. The following identity will be useful later:

$$\langle \mu_{\mathbf{C}}(x) jx, x \rangle = |\mu_{\mathbf{C}}(x)|^2 \quad (4.1.2)$$

2 The difference between the constants 2 in (4.1.1) and $\sqrt{2}$ in Proposition 4.1.7 comes from the fact that $|dz| = \sqrt{2}$ with respect to the Euclidean metric on $\mathbf{C}$.

Under the reduction of the structure group of M from SO(3) to U(1), the splitting $\mathbf{R}^3 = \mathbf{R} \oplus \mathbf{C}$ gives us $\mathfrak{su}(\not{S}) = \underline{\mathbf{R}} \oplus K^{-1}$. Here, $\not{S} = K^{1/2} \oplus K^{-1/2}$ is the spinor bundle of the spin structure $\mathfrak{s}$. Accordingly, $\mu \colon \mathfrak{S} \to \mathfrak{su}(\not{S}) \otimes \mathfrak{g}_P$ splits into the direct sum of

$$\mu_{\mathbf{R}} \colon \mathfrak{S} \to \mathfrak{g}_P \qquad \text{and} \qquad \mu_{\mathbf{C}} \colon \mathfrak{S} \to K^{-1} \otimes \mathfrak{g}_P.$$

$\mu_{\mathbf{C}}$ is holomorphic when restricted to fibers. Similarly, we have the conjugate maps

$$\mu_{\mathbf{R}} \colon \overline{\mathfrak{S}} \to \mathfrak{g}_P \qquad \text{and} \qquad \mu_{\mathbf{C}} \colon \overline{\mathfrak{S}} \to K \otimes \mathfrak{g}_P,$$

which satisfy

$$\mu_{\mathbf{R}} \circ \sigma = -\mu_{\mathbf{R}} \qquad \text{and} \qquad \mu_{\mathbf{C}} \circ \sigma = \overline{\mu_{\mathbf{C}}}.$$

Proposition 4.1.8. *Let $A_M = A(t) + b(t)\mathrm{d}t$ a connection in $\mathscr{A}_B(M, Q)$ and $\Phi = \Phi(t)$ a section in $\Gamma(M, \mathfrak{S})$. The generalized Seiberg–Witten equation (2.2.1) for (Φ, A_M) is equivalent to*

$$\begin{cases} i\left(\frac{\partial \Phi}{\partial t} + b\Phi\right) + \sqrt{2}\sigma\left(\partial_A \Phi\right) = 0, \\ \varpi\left(\frac{\partial A}{\partial t} + d_A b\right)^{0,1} = -\frac{i}{2}\mu_{\mathbf{C}}(\Phi), \\ *\varpi F_A = \mu_{\mathbf{R}}(\Phi). \end{cases} \tag{4.1.3}$$

In particular, for a S^1–invariant configuration (A, Φ) the equation simplifies to

$$\begin{cases} \partial_{AB}\Phi = 0, \\ \mu_{\mathbf{C}}(\Phi) = 0, \\ *\varpi F_A = \mu_{\mathbf{R}}(\Phi). \end{cases} \tag{4.1.4}$$

Proof. By Proposition 4.1.7, the first equation in (4.1.3) is equivalent to $\not{D}_{A_M}\Phi = 0$. The remaining two equations are obtained from the identifications

$$\mathfrak{su}(\not{S}) \cong \Lambda^0\Sigma \oplus K^{-1} \quad \text{and} \quad \mu \cong \mu_{\mathbf{R}} \oplus \mu_{\mathbf{C}} \tag{4.1.5}$$

discussed earlier. Under the decomposition

$$\Lambda^2 M = \left(\Lambda^2\Sigma\right) \oplus \left(\Lambda^1 S^1 \otimes \Lambda^1\Sigma\right) \tag{4.1.6}$$

the curvature F_{A_M} decomposes into

$$F_{A_M} = F_A + \mathrm{d}t \wedge \left(\frac{\partial A}{\partial t} + d_A b\right).$$

We need to identify the splittings (4.1.5) and (4.1.6) under the isomorphism $\Lambda^2 M \cong \mathfrak{su}(\not{S})$. For simplicity, consider the flat case $M = \mathbf{R} \times \mathbf{C}$, with coordinates t and $z = x + iy$—the general case differs from it by a conformal factor. The isomorphism $\Lambda^2\mathbf{R}^3 \cong \mathfrak{su}(2)$ is given by

$$\mathrm{d}x \wedge \mathrm{d}y \mapsto i, \quad \mathrm{d}y \wedge \mathrm{d}t \mapsto j, \quad \mathrm{d}t \wedge \mathrm{d}x \mapsto k.$$

On the other hand, $\mathfrak{su}(2)$ is identified with $\mathbf{R} \oplus \mathbf{C}$ via the map

$$ai + bj + ck \mapsto (a, b + ic).$$

Let $\alpha + dt \wedge \beta$ be a two-form on $\mathbf{R}^3$, where

$$\alpha = a dx \wedge dy, \quad \beta = b_1 dx + b_2 dy.$$

Under the identifications $\Lambda^2 \mathbf{R}^3 = \mathfrak{su}(2) = \mathbf{R} \oplus \mathbf{C}$,

$$\alpha + dt \wedge \beta \mapsto ai - b_2 j + b_1 k \mapsto (a, -b_2 + ib_1).$$

Observe that $a = *\alpha$ and $(-b_2 + ib_1) d\bar{z} = 2i\beta^{0,1}$, where $\beta^{0,1}$ is $(0,1)$-part of β. It follows that under the splittings (4.1.5) and (4.1.6) the isomorphism

$$\left(\Lambda^2 \Sigma\right) \oplus \left(\Lambda^1 S^1 \otimes \Lambda^1 \Sigma\right) \cong \Lambda^0 \Sigma \oplus K^{-1}$$

is the direct sum of the Hodge star $\Lambda^2 \Sigma \to \Lambda^0 \Sigma$ and the map $\Lambda^1 \Sigma \to \Lambda^{0,1} \Sigma$ taking a one-form β to $2i\beta^{0,1}$. Thus, $\varpi F_{A_M} = \mu(\Phi)$ is equivalent to the last two equations in (4.1.3). □

Remark 4.1.9. Since it is more common to consider holomorphic rather than aholomorphic sections, we can complete the picture by considering the conjugate bundle

$$\overline{\mathfrak{S}} = (Q \times K^{1/2}) \times_{H \times \mathrm{U}(1)} M = V \otimes K^{1/2} = \mathfrak{S} \otimes K.$$

We have the Dolbeault operators

$$\partial_A \colon \Gamma(\Sigma, \mathfrak{S}) = \Gamma(\Sigma, V \otimes K^{-1/2}) \longrightarrow \Gamma(\Sigma, V \otimes K^{1/2}) = \Gamma(\Sigma, \overline{\mathfrak{S}}),$$
$$\bar{\partial}_A \colon \Gamma(\Sigma, \overline{\mathfrak{S}}) = \Gamma(\Sigma, V \otimes K^{1/2}) \longrightarrow \Gamma(\Sigma, V \otimes K^{-1/2}) = \Gamma(\Sigma, \mathfrak{S}),$$

and the maps $\sigma \colon \overline{\mathfrak{S}} \to \mathfrak{S}$ and $\sigma \colon \mathfrak{S} \to \overline{\mathfrak{S}}$ that intertwine them:

$$\sigma \partial_A = \bar{\partial}_A \sigma.$$

Thus, σ maps aholomorphic sections of $\mathfrak{S}$ to holomorphic sections of $\overline{\mathfrak{S}}$ and vice versa. It follows from the Kähler identities that

$$\bar{\partial}_A = -\partial_A^*$$

where ∂_A^* is the formal adjoint of ∂_A.

Using σ, we can rewrite (4.1.4) as a system of equations for $\overline{\Phi} := \sigma(\Phi) \in \Gamma(\Sigma, \overline{\mathfrak{S}})$:

$$\begin{cases} \bar{\partial}_A \overline{\Phi} = 0, \\ \mu_{\mathbf{C}}(\overline{\Phi}) = 0, \\ *\varpi F_A + \mu_{\mathbf{R}}(\overline{\Phi}) = 0. \end{cases} \tag{4.1.7}$$

This is an example of a *symplectic vortex equation* discussed in [Cie+02]. The target singular symplectic space is the zero locus $\mu_{\mathbf{C}}^{-1}(0) \subset S$.

Example 4.1.10. In the case of the classical Seiberg–Witten equation, we have $G = H = \mathrm{U}(1)$, $S = \mathbf{C}^2$, and

$$\mu_{\mathbf{C}}^{-1}(0) = \{(x, y) \in \mathbf{C}^2 \mid xy = 0\}.$$

We see that $\mu_{\mathbf{C}}^{-1}(0)$ has an isolated singularity at 0.

The next theorem is the main result of this section. It assumes the existence of an irreducible solutions to the Seiberg–Witten equation.

Definition 4.1.11. A solution (Φ, A) of the Seiberg–Witten equation (2.2.1) is called *irreducible* if there exists a point $x \in M$ such that the G–stabilizer of $\Phi(x) \in \mathfrak{S}_x = S$ is trivial.

Theorem 4.1.12. *Suppose that $M = S^1 \times \Sigma$ is equipped with a set of geometric data $(g, \mathfrak{s}, Q, B)$, as in Definition 2.2.1, such that*

1. *the Riemannian metric g is the product of the standard metric on S^1 and a Riemannian metric g_Σ on Σ,*
2. *the spin structure $\mathfrak{s}$ is induced from a spin structure on Σ,*
3. *the principal K–bundle $R \to M$ is pulled back from a bundle on Σ,*
4. *the connection B is pulled back from a connection on Σ.*

If (Φ, A_M) is an irreducible solution of the Seiberg–Witten equation (2.2.1), then $Q \to M$ is pulled back from a bundle over Σ (that is: the geometric data is pulled back from Σ in the sense of Definition 4.1.1) and (Φ, A_M) is gauge-equivalent to a S^1–invariant configuration satisfying (4.1.4).

Remark 4.1.13. Theorem 4.1.12 establishes a correspondence between gauge equivalence classes of solutions of the Seiberg–Witten equation on $M = S^1 \times \Sigma$ with the hyperkähler target S and gauge equivalence classes of solutions of the symplectic vortex equation on Σ with the singular Kähler target $\mu_{\mathbf{C}}^{-1}(0)$.

Proof. Assume for simplicity that the algebraic data from Definition 2.1.7 is so that $G = H$; the general proof is the same after adjusting the notation. In this case, the flavor symmetry group K is trivial, and so are the bundle R and connection B appearing in the geometric data from Definition 2.2.1. Identify S^1 with $[0, 1]$ with the endpoints glued together. Pull back the data on $M = S^1 \times \Sigma$ to one-periodic data on $[0, 1] \times \Sigma$. Since $[0, 1] \times \Sigma$ is homotopy equivalent to Σ, there is a principal G–bundle $Q_\Sigma \to \Sigma$ and a gauge transformation $g \in \mathscr{G}(\Sigma, Q_\Sigma)$ such that Q is the quotient of $[0, 1] \times Q_\Sigma$ by the relation $(0, p) \sim (1, g(p))$. The isomorphism class of Q depends only on the homotopy class of g. Similarly, $\mathfrak{S} = (Q \times S) \times_{G \times SU(2)} M$ over M is obtained from pulling back $\mathfrak{S}_\Sigma = (Q_\Sigma \times K^{-1/2})_{G \times U(1)} M$ to $[0, 1] \times \Sigma$ and identifying the fibers over 0 and 1 using g. As before there is an anti-linear map $\sigma \colon \mathfrak{S}_\Sigma \otimes K \to \mathfrak{S}_\Sigma$.

For $A_M \in \mathscr{A}_B(M, Q)$ and $\Phi \in \Gamma(M, \mathfrak{S})$ we have

$$A_M = A(t) + b(t)\mathrm{d}t, \quad \Phi = \Phi(t),$$

where $A(t), b(t)$, and $\Phi(t)$ are families of connections and sections on Σ, as discussed earlier. The only difference now is that the families are periodic with respect to the action of g:

$$A(1) = g\left(A(0)\right), \quad b(1) = g\left(b(0)\right), \quad \Phi(1) = g\left(\Phi(0)\right).$$

Define a gauge transformation h over $[0,1] \times \Sigma$ by

$$h(t) = \exp\left(\int_0^t b(s) ds\right). \tag{4.1.8}$$

h does not necessarily descend to an automorphism of $Q \to M$; this happens if and only if $h(1) = \mathrm{Ad}_g(h_0) = \mathrm{id}$. In any case, h is well-defined over $[0,1] \times \Sigma$ and the new connection

$$C := h(A_M) = A_M - h^{-1} d_{A_M} h = h(t)\,(A(t))$$

does not have a $\mathrm{d}t$ part—it is in a *temporal gauge*. Thus, it is identified with a path of connections $\{C(t)\}_{t\in[0,1]}$ on Q_Σ satisfying $C(1) = h(1)g(C(0))$. Likewise, we identify the section $h(\Phi)$ with a path $\{\Phi(t)\}_{t\in[0,1]}$ of sections of $\mathfrak{S}_\Sigma \to \Sigma$ satisfying $\Phi(1) = h(1)g(\Phi(0))$.

By Proposition 4.1.8, the Seiberg–Witten equation for (Φ, C) is equivalent to

$$\begin{cases} i\frac{\partial \Phi}{\partial t} + \sqrt{2}\sigma\left(\partial_C \Phi\right) = 0, \\ \left(\frac{\partial C}{\partial t}\right)^{0,1} = -\frac{i}{2}\mu_{\mathbf{C}}(\Phi), \\ *F_C = \mu_{\mathbf{R}}(\Phi). \end{cases} \tag{4.1.9}$$

Differentiating the first equation with respect to t and using (4.1.9), we obtain

$$\begin{aligned} 0 &= i\frac{\partial^2 \Phi}{\partial t^2} + \sqrt{2}\sigma\left\{\left(\frac{\partial C}{\partial t}\right)^{1,0}\Phi + \partial_C\left(\frac{\partial \Phi}{\partial t}\right)\right\} \\ &= i\frac{\partial^2 \Phi}{\partial t^2} + \frac{\sqrt{2}}{2}\sigma i\overline{\mu_{\mathbf{C}}}(\Phi)\Phi + 2\sigma i \partial_C \sigma \partial_C \Phi \\ &= i\frac{\partial^2 \Phi}{\partial t^2} - \frac{\sqrt{2}}{2} i \mu_{\mathbf{C}}(\Phi)\sigma\Phi - 2i\sigma\partial_C\sigma\partial_C\Phi. \end{aligned}$$

We have used the anti-linearity of σ and the fact that $\partial C/\partial t$ is a real $\mathfrak{g}$–valued one-form, so its $(1,0)$ part is conjugate to the $(0,1)$ part. Multiplying the obtained identity by i and taking the pointwise inner product with Φ yields

$$0 = \left\langle -\frac{\partial^2 \Phi}{\partial t^2}, \Phi \right\rangle + \frac{\sqrt{2}}{2}\left\langle \mu_{\mathbf{C}}(\Phi)\sigma\Phi, \Phi\right\rangle + 2\left\langle \sigma\partial_C\sigma\partial_C\Phi, \Phi\right\rangle. \tag{4.1.10}$$

By formula (4.1.2) the second term simplifies to $\sqrt{2}/2|\mu_{\mathbf{C}}(\Phi)|^2$. Remark 4.1.9 implies that

$$\sigma\partial_C\sigma\partial_C = \bar{\partial}_C\left(\sigma^2\right)\partial_C = -\bar{\partial}_C\partial_C = \partial_C^*\partial_C.$$

We conclude that

$$\langle \sigma\partial_C\sigma\partial_C\Phi, \Phi\rangle_{L^2(\Sigma)} = \langle \partial_C^*\partial_C\Phi, \Phi\rangle_{L^2(\Sigma)} = \|\partial_C\Phi\|^2_{L^2(\Sigma)}.$$

For a fixed value of t integration of (4.1.10) over Σ yields

$$0 = \int_\Sigma \left\langle -\frac{\partial^2\Phi}{\partial t^2}, \Phi\right\rangle \mathrm{vol}_\Sigma + \frac{\sqrt{2}}{2}\|\mu_{\mathbf{C}}(\Phi)\|^2_{L^2(\Sigma)} + 2\|\partial_C\Phi\|^2_{L^2(\Sigma)}.$$

Integrate the last equality by parts with respect to $t \in [0,1]$. The boundary terms vanish because Φ is periodic up to the action of $h(1)g$ which preserves the inner product. We obtain

$$0 = \left\| \frac{\partial \Phi}{\partial t} \right\|_{L^2}^2 + \frac{\sqrt{2}}{2} \left\| \mu_{\mathbf{C}}(\Phi) \right\|_{L^2}^2 + 2 \left\| \partial_C \Phi \right\|_{L^2}^2,$$

which shows that

$$\frac{\partial \Phi}{\partial t} = 0, \quad \frac{\partial C}{\partial t} = 0.$$

Thus, the families $C(t) = C$ and $\Phi(t) = \Phi$ are constant and

$$C = C(1) = k\left(C(0)\right) = k(C), \quad \Phi = \Phi(1) = k\left(\Phi(0)\right) = k(\Phi)$$

for the gauge transformation $k = h(1)g$ over Σ. The first equality implies $d_C k = 0$, so k is covariantly constant. On the other hand, by irreducibility, there exists a point $x \in \Sigma$ such that the G–stabiliser of $\Phi(x)$ is trivial. Hence, $k(x) = \mathrm{id}$, so $k = \mathrm{id}$ everywhere and $g = h(1)^{-1}$. The path $h(t)^{-1}$ is a homotopy of gauge transformations connecting g with $h(0)^{-1} = \mathrm{id}$ and so $Q \to M$ is pulled back from $Q_\Sigma \to \Sigma$. In particular, we could have chosen $g = \mathrm{id}$, then $h(1) = \mathrm{id}$ and h descends to a gauge transformation of Q mapping (A_M, Φ) to the S^1–invariant solution (C, Φ). By Proposition 4.1.8, (C, Φ) satisfies equation (4.1.4). □

Remark 4.1.14. Much of this discussion can be extended to the setting when S is a hyperkähler manifold with an isometric $\mathrm{Sp}(1)$–action, which rotates the sphere of complex structures on S. The Dirac operator $\not{D}_A$ and equation (2.2.1) have natural generalizations [Hay12; Hay14a]. For $M = S^1 \times \Sigma$ one introduces the non-linear Dolbeault operator $\bar{\partial}_A$ as in [Cie+02] so that Proposition 4.1.7 and Proposition 4.1.8 hold. However, our proof of Theorem 4.1.12 makes use of the vector space structure on S and does not immediately generalize to the non-linear setting. We expect the result to be true, but in the proof one should use the Weitzenböck formula for non-linear Dirac operators [Tau99; Pid04b; Cal15].

4.2 ADHM BUNDLES ON RIEMANN SURFACES

In Section 2.9 we introduced the notion of an ADHM monopole on a 3–manifold. In this section, we apply Theorem 4.1.12 to characterize ADHM monopoles on $M = S^1 \times \Sigma$ in terms of the complex geometry of Σ.

We will consider the $\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation on M under the assumption that the geometric data required to write down the equation is pulled-back from Σ. This is similar to the situation described in Definition 4.1.1. However, in order to discuss the $\mathrm{ADHM}_{r,k}$ Seiberg–Witten equation, it is convenient to use $\mathrm{spin}^{\mathrm{U}(k)}$ structures rather than spin structures. Specifically, assume that we are given a set of geometric data as in Definition 2.9.5 such that

1. g is a product Riemannian metric,
2. E and the connection B are pulled-back from Σ, and
3. V and the connection C are pulled-back from a $\mathrm{U}(2)$–bundle with a connection on Σ such that $\Lambda^2_{\mathbf{C}} V \cong K_\Sigma$ as bundles with connections.

Proposition 4.2.1. *If the above three conditions hold and (Ψ, ξ, A) is an irreducible solution of the ADHM$_{r,k}$ Seiberg–Witten equation (2.9.3), then the spin$^{\mathrm{U}(k)}$ structure $\mathfrak{w}$ is pulled-back from a spin$^{\mathrm{U}(k)}$ structure on Σ and (Ψ, ξ, A) is gauge-equivalent to a configuration pulled-back from Σ.*

Recall that any spin$^{\mathrm{U}(k)}$ structure on a 3–manifold is obtained from a spin structure and a $\mathrm{U}(k)$–bundle as in Example 2.9.4. Thus, the above result can be formulated in the language of spin structures and is a special case of Theorem 4.1.12.

In the situation of Proposition 4.2.1, equation (2.9.3) reduces to a non-abelian vortex equation on Σ. Recall that a choice of a spin$^{\mathrm{U}(k)}$ structure on Σ is equivalent to a choice of a $\mathrm{U}(k)$–bundle $H \to \Sigma$. Consequently, A can be seen as a connection on H. The corresponding spinor bundles are

$$\mathfrak{g}_H = \mathfrak{u}(H) \quad \text{and} \quad W = H \oplus T^*\Sigma^{0,1} \otimes H.$$

Proposition 4.2.2. *Let (A, Ψ, ξ) be a configuration pulled-back from Σ. Under the splitting $W = H \oplus T^*\Sigma^{0,1} \otimes H$ we have $\Psi = (\psi_1, \psi_2^*)$ where*

$$\begin{aligned} \psi_1 &\in \Gamma(\Sigma, \mathrm{Hom}(E, H)), \\ \psi_2 &\in \Omega^{1,0}(\Sigma, \mathrm{Hom}(H, E)), \quad \textit{and} \\ \xi &\in \Gamma(\Sigma, V \otimes \mathrm{End}(H)). \end{aligned}$$

Equation (2.9.3) for (A, Ψ, ξ) is equivalent to

$$\begin{gathered} \bar\partial_{A,B}\psi_1 = 0, \quad \bar\partial_{A,B}\psi_2 = 0, \quad \bar\partial_{A,C}\xi = 0, \\ [\xi \wedge \xi] + \psi_1\psi_2 = 0, \quad \textit{and} \\ i * F_A + [\xi \wedge \xi^*] + \psi_1\psi_1^* - *\psi_2^*\psi_2 = 0. \end{gathered} \tag{4.2.1}$$

In the second equation we use the isomorphism $\Lambda^2_{\mathbf{C}} V \cong K_\Sigma$ so that the left-hand side is a section of $\Omega^{1,0}(\Sigma, \mathrm{End}(H))$. In the third equation we contract V with V^* so that the left-hand side is a section of $i\mathfrak{u}(H)$.

This follows from Proposition 4.1.8, Remark 4.1.9, and the complex description of the hyperkähler moment map appearing in the ADHM construction, cf. [Nak99, Chapter 2] and [DW17b, Appendix D]

We can also perturb (4.2.1) by $\tau \in \mathbf{R}$ and $\theta \in H^0(\Sigma, K_\Sigma)$:

$$\begin{gathered} \bar\partial_{A,B}\psi_1 = 0, \quad \bar\partial_{A,B}\psi_2 = 0, \quad \bar\partial_{A,C}\xi = 0, \\ [\xi \wedge \xi] + \psi_1\psi_2 = \theta \otimes \mathrm{id}, \quad \text{and} \\ i * F_A + [\xi \wedge \xi^*] - \psi_1\psi_1^* + *\psi_2^*\psi_2 = \tau\, \mathrm{id}. \end{gathered} \tag{4.2.2}$$

There is a Hitchin–Kobayashi correspondence between gauge-equivalence classes of solutions of (4.2.2) and isomorphism classes of certain holomorphic data on Σ. Let $\mathscr{E} = (E, \bar\partial_B)$ and $\mathscr{V} = (V, \bar\partial_C)$ be the holomorphic bundles induced from the unitary connections on E and V.

Definition 4.2.3. An *ADHM bundle* with respect to $(\mathscr{E}, \mathscr{V}, \theta)$ is a quadruple

$$(\mathscr{H}, \psi_1, \psi_2, \xi)$$

consisting of:

- a rank k holomorphic vector bundle $\mathscr{H} \to \Sigma$,
- $\psi_1 \in H^0(\Sigma, \mathrm{Hom}(\mathscr{E}, \mathscr{H}))$,
- $\psi_2 \in H^0(\Sigma, K_\Sigma \otimes \mathrm{Hom}(\mathscr{H}, \mathscr{E}))$, and
- $\xi \in H^0(\Sigma, \mathscr{V} \otimes \mathrm{End}(\mathscr{H}))$

such that

$$[\xi \wedge \xi] + \psi_1\psi_2 = \theta \otimes \mathrm{id} \in H^0(\Sigma, K_\Sigma \otimes \mathrm{End}(\mathscr{H})).$$

Definition 4.2.4. For $\delta \in \mathbf{R}$, the δ–*slope* of an ADHM bundle $(\mathscr{H}, \psi_1, \psi_2, \xi)$ is

$$\mu_\delta(\mathscr{H}) := \frac{2\pi}{\mathrm{vol}(\Sigma)} \frac{\deg \mathscr{H}}{\mathrm{rk}\, \mathscr{H}} + \frac{\delta}{\mathrm{rk}\, \mathscr{H}}.$$

The *slope* of $\mathscr{H}$ is $\mu(\mathscr{H}) := \mu_0(\mathscr{H})$.

Definition 4.2.5. Let $\delta \in \mathbf{R}$. An ADHM bundle $(\mathscr{H}, \psi_1, \psi_2, \xi)$ is δ–*stable* if it satisfies the following conditions:

1. If $\delta > 0$, then $\psi_1 \neq 0$ and if $\delta < 0$, then $\psi_2 \neq 0$.
2. If $\mathscr{G} \subset \mathscr{H}$ is a proper ξ–invariant holomorphic subbundle such that $\mathrm{im}\, \psi_1 \subset \mathscr{G}$, then $\mu_\delta(\mathscr{G}) < \mu_\delta(\mathscr{H})$.
3. If $\mathscr{G} \subset \mathscr{H}$ is a proper ξ–invariant holomorphic subbundle such that $\mathscr{G} \subset \ker \psi_2$, then $\mu(\mathscr{G}) < \mu_\delta(\mathscr{H})$.

We say that $(\mathscr{H}, \psi_1, \psi_2, \xi)$ is δ–*polystable* if there exists a ξ–invariant decomposition $\mathscr{H} = \bigoplus_i \mathscr{G}_i \bigoplus_j \mathscr{I}_j$ such that:

1. $\mu_\delta(\mathscr{G}_i) = \mu_\delta(\mathscr{H})$ for every i and the restrictions of (ψ_1, ψ_2, ξ) to each $\mathscr{G}_i$ define a δ stable ADHM bundle, and
2. $\mu(\mathscr{I}_j) = \mu_\delta(\mathscr{H})$ for every j, the restrictions of ψ_1, ψ_2 to each $\mathscr{I}_j$ are zero, and there exist no ξ–invariant proper subbundle $\mathscr{J} \subset \mathscr{I}_i$ with $\mu(\mathscr{J}) < \mu(\mathscr{I}_j)$.

In the proposition below we fix δ and the topological type of $\mathscr{H}$, and set $\tau = \mu_\delta(\mathscr{H})$.

Proposition 4.2.6. *Let (A, ψ_1, ψ_2, ξ) be a solution of (4.2.2). Denote by $\mathscr{H}$ the holomorphic vector bundle $(H, \bar\partial_A)$. Then $(\mathscr{H}, \psi_1, \psi_2, \xi)$ is a δ–polystable ADHM bundle. Conversely, every δ–polystable ADHM bundle arises in this way from a solution to (4.2.2) which is unique up to gauge equivalence.*

Proof. A standard calculation going back to [Don83] shows that (4.2.2) implies δ–polystability. The difficult part is showing that every δ–polystable ADHM bundle admits a compatible unitary connection solving the third equation of (4.2.2), unique up to gauge equivalence. This is a special case of the main result of [ACP03, Theorem 3.1], with the minor difference that the connections on the bundles E and V are fixed and not part of a solution. The necessary adjustment in the proof is discussed in a similar setting in [BPR03]. □

Remark 4.2.7. Stable ADHM bundles on Riemann surfaces were studied extensively by Diaconescu [Dia12b; Dia12a], who related them to the *local stable pair invariants* of the non-compact Calabi–Yau 3–fold $\mathscr{V}$. This suggests that there is a relationship between ADHM monopoles and enumerative invariants in algebraic geometry. We elaborate on this relationship in [DW17b, Section 7].

4.3 MONOPOLES WITH TWO SPINORS

We specialize further to the Seiberg–Witten equation with two spinors, which was introduced in Section 3.2, and which is closely related to the $\mathrm{ADHM}_{2,1}$ Seiberg–Witten equation (see the discussion at the end of Section 2.9). In this section, we give two constructions of the moduli space of monopoles with two spinors on $M = S^1 \times \Sigma$, one using gauge theory and one using complex geometry, and we prove that they are isomorphic as real analytic spaces; see Theorem 4.3.6 below. The results of this section are inspired by work of Bryan and Wentworth [BW96].

Let $M = S^1 \times \Sigma$. Let $E \to M$ be an $\mathrm{SU}(2)$–bundle pulled back from a bundle on Σ. Recall that the Seiberg–Witten equation with two spinors depends on the choice of a parameter $\mathbf{p} = (g, B) \in \mathscr{M}et(M) \times \mathscr{A}(E)$ and a closed 2–form $\eta \in \mathscr{Z}$; see Definition 3.1.5 and equation (3.2.1).

Definition 4.3.1. Let $\mathscr{M}et(\Sigma)$ be the space of Riemannian metrics on Σ. Define the *space of parameters pulled back from* Σ by

$$\mathscr{P}_\Sigma := \mathscr{M}et(\Sigma) \times \mathscr{A}(\Sigma, E).$$

Moreover, let $\mathscr{Z}_\Sigma := \Omega^2(\Sigma, i\mathbf{R})$. We have $\mathscr{P}_\Sigma \subset \mathscr{P}$ and $\mathscr{Z}_\Sigma \subset \mathscr{Z}$.

The next result is a special case of Proposition 4.2.2.

Proposition 4.3.2. *If (Ψ, A) is a solution of the Seiberg–Witten equation with two spinors (3.2.2) with respect to $(\mathbf{p}, \eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$, then*

1. *the spin^c structure $\mathfrak{w}$ is pulled back from a spin^c structure on Σ,*
2. *(Ψ, A) is gauge equivalent to a configuration pulled back from Σ.*

Concretely, there exist a spin structure $K^{1/2}$ on Σ and a Hermitian line bundle $L \to \Sigma$ such that the spinor bundle of $\mathfrak{w}$ is pulled back from

$$W = (K^{1/2} \otimes L) \oplus (K^{-1/2} \otimes L).$$

After a gauge transformation,

$$\begin{aligned} &A \in \mathscr{A}(\Sigma, L), \quad \text{and} \\ &\Psi = (\alpha, \bar{\beta}), \end{aligned}$$

with

$$\begin{aligned} &\alpha \in \Gamma(\Sigma, E^* \otimes L \otimes K^{1/2}) \\ &\beta \in \Gamma(\Sigma, E \otimes L^* \otimes K^{1/2}) \end{aligned}$$

and the following equations are satisfied

$$\begin{aligned} \bar{\partial}_{A,B}\alpha &= 0, \\ \bar{\partial}_{A,B}\beta &= 0, \\ \alpha\beta &= 0, \\ i * F_A &= |\alpha|^2 - |\beta|^2 + i * \eta. \end{aligned} \tag{4.3.1}$$

Here, $\alpha\beta \in \Gamma(\Sigma, K)$ is obtained from $\alpha \otimes \beta$ using the contractions $E \otimes E^* \to \mathbf{C}$ and $L \otimes L^* \to \mathbf{C}$.

Remark 4.3.3. An attentive reader will notice that we have again switched from the language of spin^c structures to that of spin structures. The change is only cosmetic, as every spin^c structure on a Riemann surface can be obtained from a spin structure and a Hermitian line bundle. Choosing a spin structure $K^{1/2}$ on Σ brings certain symmetry to the roles of α and β, which will be convenient later, when discussing Fueter sections.

In the situation described in Proposition 4.3.2, the moduli space $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ has the following description. Let

$$\mathscr{C}_\Sigma = \mathscr{A}(\Sigma, L) \times \Gamma(\Sigma, E^* \otimes L \otimes K^{1/2}) \times \Gamma(\Sigma, E \otimes L^* \otimes K^{1/2})$$

Consider the subspace of $\mathscr{C}_\Sigma$ consisting triples (A, α, β) satisfying equations (4.3.1). The gauge group $\mathscr{G}(\Sigma, L) = C^\infty(\Sigma, S^1)$ acts on this subspace and the action is free whenever $(\alpha, \beta) \neq (0,0)$. By Proposition 4.3.2 and Proposition 4.1.6, the quotient is homeomorphic to $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$.

Remark 4.3.4. To simplify the notation, assume for the remaining part of this section that the spin^c structure $\mathfrak{w}$ and parameters $(\mathbf{p}, \eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$ are fixed and we write simply

$$\mathfrak{M} := \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta).$$

The next result extends the work of Bryan and Wentworth [BW96] who described monopoles with multiple spinors on Kähler surfaces under the assumption that the background bundle E is trivial and B is the product connection. Before stating the theorem, we introduce the following notation:

$$d = \deg(L) := \langle c_1(L), [\Sigma] \rangle \qquad \text{and} \qquad \tau := \int_\Sigma \frac{i\eta}{2\pi}.$$

If $d - \tau < 0$, then the last equation of (4.3.1) forces α to be non-zero for

$$0 = \int_\Sigma \left\{ iF_A - i\eta + (|\alpha|^2 - |\beta|^2)\mathrm{vol}_\Sigma \right\} = 2\pi(d-\tau) + \|\alpha\|^2_{L^2} - \|\beta\|^2_{L^2}.$$

Likewise, if $d - \tau > 0$, then β must be non-zero. In both cases there are no reducible solutions. When $d - \tau = 0$, either both α and β are non-zero or both of them vanish yielding a reducible solution.

Recall that every unitary connection on a vector bundle over a Riemann surface equips the underlying vector bundle with a holomorphic structure. In particular, $K^{1/2}$ and K are naturally holomorphic line bundles.

Definition 4.3.5. Denote by $\mathscr{E} \to \Sigma$ the holomorphic $\mathrm{SL}(2, \mathbf{C})$–bundle obtained from E^* equipped with the dual connection B^*.

Theorem 4.3.6. *If $d - \tau < 0$, then $\mathfrak{M}$ is isomorphic as real analytic spaces to the moduli space $\mathfrak{M}^{\mathrm{hol}}$ of triples $(\mathscr{L}, \alpha, \beta)$ consisting of*

- *a degree d holomorphic line bundle $\mathscr{L} \to \Sigma$,*
- *holomorphic sections*

$$\alpha \in H^0(\Sigma, \mathscr{E} \otimes \mathscr{L} \otimes K^{1/2}) \quad \text{and} \quad \beta \in H^0(\Sigma, \mathscr{E}^* \otimes \mathscr{L}^* \otimes K^{1/2})$$

satisfying $\alpha \neq 0$ and $\alpha\beta = 0 \in H^0(\Sigma, K)$.

Two such triples $(\mathscr{L}, \alpha, \beta)$ and $(\mathscr{L}', \alpha', \beta')$ correspond to the same point in the $\mathfrak{M}^{\mathrm{hol}}$ if there is a holomorphic isomorphism $\mathscr{L} \to \mathscr{L}'$ mapping α to α' and β to β'.

The statement still holds when $d - \tau \geqslant 0$ with the difference that for $d - \tau > 0$ it is β instead of α that is required to be non-zero and for $d - \tau = 0$ both α and β are required to be non-zero.

4.3.1 *Construction of the holomorphic moduli space*

We will now explain how to construct $\mathfrak{M}^{\mathrm{hol}}$ analytically. A related construction was considered in [FM99, Section 1]. The first three equations in (4.3.1),

$$\begin{cases} \bar{\partial}_{AB}\alpha = 0, \\ \bar{\partial}_{AB}\beta = 0, \\ \alpha\beta = 0, \end{cases} \tag{4.3.2}$$

are invariant under the action of the *complexified gauge group* $\mathscr{G}^{\mathbf{C}}(\Sigma, L) := C^\infty(\Sigma, \mathbf{C}^*)$ of complex automorphisms of L. The action of $g\colon \Sigma \to \mathbf{C}^*$ on $(A, \alpha, \beta) \in \mathscr{C}_\Sigma$ is given by

$$g(A, \alpha, \beta) = \left(A + \overline{g}^{-1}\partial\overline{g} - g^{-1}\bar{\partial}g, g\alpha, g^{-1}\beta\right).$$

In terms of the associated Dolbeault operators we have

$$\begin{aligned} \bar{\partial}_{g(A)B} &= g\bar{\partial}_{BA}g^{-1} \quad \text{on } \Gamma(\Sigma, E^* \otimes L \otimes K^{1/2}), \\ \bar{\partial}_{g(A)B} &= g^{-1}\bar{\partial}_{BA}g \quad \text{on } \Gamma(\Sigma, E \otimes L^* \otimes K^{1/2}). \end{aligned}$$

Definition 4.3.7. Consider the subspace of $\mathscr{C}_\Sigma$ consisting of triples (A, α, β) satisfying equations (4.3.2) and subject to the condition

$$\begin{cases} \alpha \neq 0 & \text{if } d - \tau < 0, \\ \beta \neq 0 & \text{if } d - \tau > 0, \\ \alpha \neq 0 \text{ and } \beta \neq 0 & \text{if } d - \tau = 0. \end{cases}$$

We define $\mathfrak{M}^{\mathrm{hol}}$ to be the quotient of this subspace by the action of $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$. It is clear that the points of $\mathfrak{M}^{\mathrm{hol}}$ parametrize the isomorphism classes of triples $(\mathscr{L}, \alpha, \beta)$ considered in Theorem 4.3.6.

Remark 4.3.8. The moduli space $\mathfrak{M}^{\mathrm{hol}}$ depends on the conformal class of the metric g on Σ, the holomorphic $\mathrm{SL}(2, \mathbf{C})$–bundle $\mathscr{E}$, the degree d of L, and the sign of $d - \tau$, that is:

$$\mathfrak{M}^{\mathrm{hol}} = \mathfrak{M}^{\mathrm{hol}}_d(g, \mathscr{E}, \tau) = \mathfrak{M}^{\mathrm{hol}}_d(\mathbf{p}, \eta).$$

The data $(g, \mathscr{E}, \tau)$ can be recovered from the data $\mathfrak{w}$, $\mathbf{p}$, η, required to write down the Seiberg–Witten equation with two spinors; schematically:

$$\begin{aligned} \mathfrak{w} &\longleftrightarrow d \\ \mathbf{p} &\longleftrightarrow (g, \mathscr{E}) \\ \eta &\longleftrightarrow \tau. \end{aligned}$$

$\mathfrak{M}^{\mathrm{hol}}$ is metrizable, second countable, and has a natural complex analytic structure given by local Kuranishi models, as in Remark 3.2.3. The discussion is almost the same as that for the Seiberg–Witten equation (see Section 2.3 and Section 3.2), so we only outline the details. To set up the Fredholm theory, consider the modified equation

$$\begin{cases} \bar{\partial}_{AB}\alpha + if\bar{\beta} = 0, \\ \bar{\partial}_{AB}\beta - if\bar{\alpha} = 0, \\ \alpha\beta + \partial f = 0. \end{cases} \tag{4.3.3}$$

A solution of (4.3.3) is a quadruple (A, α, β, f) where A, α, and β are as before and $f \in C^\infty(\Sigma, \mathbf{C})$. The equation is elliptic modulo the action of $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$. The next result is proved by integration by parts.

Proposition 4.3.9. *If (A, α, β, f) is a solution of (4.3.3) with $(\alpha, \beta) \neq 0$, then $f = 0$.*

Using the linearization of (4.3.3) together with the complex Coulomb gauge fixing we represent $\mathfrak{M}^{\mathrm{hol}}$ as the zero set of a Fredholm section. The local structure of the moduli space is encoded in the elliptic complex at a solution $(A, \alpha, \beta, 0)$:

$$\Omega^0(\mathbf{C}) \xrightarrow{G^c_{A,\alpha,\beta}} \Omega^{0,1} \oplus \Gamma(E^* \otimes L \otimes K^{1/2}) \oplus \Gamma(E \otimes L^* \otimes K^{1/2}) \oplus \Omega^0(\mathbf{C}) \longrightarrow$$

$$\xrightarrow{F_{A,\alpha,\beta}} \Gamma(E^* \otimes L \otimes K^{-1/2}) \oplus \Gamma(E \otimes L^* \otimes K^{-1/2}) \oplus \Omega^{1,0}$$

where $G^c_{A,\alpha,\beta}$ is the linearized action of the complexified gauge group

$$G^c_{A,\alpha,\beta}(h) = (-\bar{\partial}h, h\alpha, -h\beta, 0),$$

and $F_{A,\alpha,\beta}$ is the linearization of equations (4.3.3)

$$F_{A,\alpha,\beta}(a^{0,1}, u, v, t) = \begin{pmatrix} \bar{\partial}_{AB}u + a^{0,1}\alpha + it\bar{\beta} \\ \bar{\partial}_{AB}v - a^{0,1}\beta - it\bar{\alpha} \\ u\beta + \alpha v + \partial t \end{pmatrix}.$$

Even though the map given by the left-hand side of (4.3.3) is not holomorphic, its derivative $F_{A,\alpha,\beta}$ at a solution $(A, \alpha, \beta, 0)$ is complex linear and so the cohomology groups $H^0_{A,\alpha,\beta}$, $H^1_{A,\alpha,\beta}$, $H^2_{A,\alpha,\beta}$ of the elliptic complex are complex vector spaces. If the solution is irreducible, then $H^0_{A,\alpha,\beta} = 0$. We are left with complex vector spaces $H^1_{A,\alpha,\beta}$ and $H^2_{A,\alpha,\beta}$ of the same dimension. They have the following description.

Lemma 4.3.10. *Let (A, α, β, f) be a solution of (4.3.2) with $(\alpha, \beta) \neq 0$ and $f = 0$. Then the deformation space $H^1_{A,\alpha,\beta}$ is the quotient of the space of solutions*

$$(a^{0,1}, u, v) \in \Omega^{0,1}(\mathbf{C}) \oplus \Gamma(E^* \otimes L \otimes K^{1/2}) \oplus \Gamma(E \otimes L^* \otimes K^{1/2}),$$

$$\begin{cases} \bar{\partial}_{AB}u + a^{0,1}\alpha = 0, \\ \bar{\partial}_{AB}v - a^{0,1}\beta = 0, \\ u\beta + \alpha v = 0. \end{cases}$$

by the subspace generated by $(-\bar{\partial}h, h\alpha, -h\beta)$ for $h \in \Omega^0(\mathbf{C})$. The obstruction space $H^2_{A,\alpha,\beta}$ is canonically isomorphic to the dual space $(H^1_{A,\alpha,\beta})^*$ as complex vector spaces.

The analytic structure on a neighbourhood of $[A, \alpha, \beta]$ in $\mathfrak{M}^{\mathrm{hol}}$ is induced from a Kuranishi map $\kappa \colon H^1_{A,\alpha,\beta} \to H^2_{A,\alpha,\beta}$. Since the derivative of $F_{A,\alpha,\beta}$ is complex linear at a solution, κ can be taken to be complex analytic which shows that $\mathfrak{M}^{\mathrm{hol}}$ is a complex analytic space.

4.3.2 *Isomorphism of real analytic spaces*

In the remaining part of this section, we construct an isomorphism $\mathfrak{M} \cong \mathfrak{M}^{\mathrm{hol}}$, thus proving Theorem 4.3.6. First, we construct a homeomorphism $\mathfrak{M} \cong \mathfrak{M}^{\mathrm{hol}}$. Since monopoles with two spinors are essentially solutions of the $\mathrm{ADHM}_{2,1}$ Seiberg–Witten equation, the existence of such a homeomorphism is a special case of the Hitchin–Kobayashi correspondence for ADHM monopoles; see Proposition 4.2.6. However, here we present a more explicit construction of this homeomorphism, which will be useful later, when we discuss compactifications of the moduli spaces.

Since equation (4.3.2) is part of (4.3.1) and $\mathscr{G}(\Sigma, L)$ is a subgroup of $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$, every point of $\mathfrak{M}$ gives rise to a point in $\mathfrak{M}^{\mathrm{hol}}$.

Proposition 4.3.11. *The natural map $\mathfrak{M} \to \mathfrak{M}^{\mathrm{hol}}$ is a homeomorphism.*

The proof relies on a generalization of a classical theorem of Kazdan and Warner. We refer to [BW96] for the proof.

Lemma 4.3.12 (Bryan and Wentworth [BW96, Lemma 3.4])**.** *Let X be a compact Riemannian manifold and let P, Q, and w be smooth functions on X with P and Q non-negative, and*

$$\int_X P - Q > 0, \qquad \int_X w > 0.$$

Then the equation

$$\Delta u + Pe^u - Qe^{-u} = w$$

has a unique solution $u \in C^\infty(X)$.

The same statement holds when $\int w = 0$ and both P and Q are not identically zero (without the assumption on the sign of $\int P - Q$).

Proof of Proposition 4.3.11. It is clear that the map $\mathfrak{M} \to \mathfrak{M}^{\mathrm{hol}}$ is continuous, so it remains to construct a continuous inverse $\mathfrak{M}^{\mathrm{hol}} \to \mathfrak{M}$. Let (A, α, β) be a solution of (4.3.2). As in [BW96], we seek $h \in \mathscr{G}^{\mathbf{C}}(\Sigma, L)$ such that $h(A, \alpha, \beta) = (A', \alpha', \beta')$ satisfies also the third equation of (4.3.1). We can assume $h = e^f$ for $f \colon \Sigma \to \mathbf{R}$. We have

$$(A', \alpha', \beta') = (A - \bar\partial f + \partial f, e^f \alpha, e^{-f} \beta)$$

so the curvature of A' is

$$F_{A'} = F_A - 2\partial\bar\partial f = F_A - i * \Delta f,$$

where Δ is the positive definite Hodge Laplacian. Thus, (4.3.1) for (A', α', β') is equivalent to

$$\begin{aligned} 0 &= i * F_{A'} + |\alpha'|^2 - |\beta'|^2 - i\eta \\ &= \Delta f + e^{2f}|\alpha|^2 - e^{-2f}|\beta|^2 + i(*F_A - \eta). \end{aligned}$$

Assume $d - \tau < 0$ and set $P = |\alpha|^2$, $Q = |\beta|^2$ and $w = -i * (F_A - \eta)$. We need to solve

$$\Delta f + Pe^{2f} - Qe^{-2f} = w. \tag{4.3.4}$$

If $d - \tau < 0$, then α is assumed to be non-zero. After applying a gauge transformation of the form $h = e^C$ for C constant, we may assume that

$$\int_\Sigma (P - Q) = \int_\Sigma |\alpha|^2 - |\beta|^2 > 0.$$

Moreover, we have

$$\int_\Sigma w = -\int_\Sigma (iF_A - \eta) = -2\pi(d - \tau) > 0.$$

The hypotheses of Lemma 4.3.12 are satisfied and there is a unique solution $f \in C^\infty(\Sigma)$ to (4.3.4). This shows that there exists $h \in \mathscr{G}^{\mathbf{C}}(\Sigma, L)$, unique up to an element of $\mathscr{G}(\Sigma, L)$, mapping (A, α, β) to a solution of (4.3.1). The proof is similar in the cases $d - \tau > 0$ or $d - \tau = 0$.

This gives us an inverse to $\mathfrak{M} \to \mathfrak{M}^{\text{hol}}$; it remains to show that it is continuous. Let $[A_i, \alpha_i, \beta_i]$ be a convergent sequence of points in $\mathfrak{M}^{\text{hol}}$. Let $(A_i', \alpha_i', \beta_i')$ be the corresponding solutions of (4.3.1). There is a sequence $h_i = u_i e^{f_i}$ such that $h_i(A_i', \alpha_i', \beta_i')$ converges in $\mathscr{C}_\Sigma$. The functions f_i satisfy (4.3.4) with coefficients P_i, Q_i, w_i converging in $C^\infty(\Sigma)$. It follows from the proof of [Doa17a, Proposition 3.1] that for every k there is a C^k bound for f_i, independent of i. By the Arzelà–Ascoli theorem, after passing to a subsequence, f_i converges in $C^\infty(\Sigma)$. It follows that $[A_i', \alpha_i', \beta_i']$ converges in $\mathfrak{M}$, which proves the continuity of $\mathfrak{M}^{\text{hol}} \to \mathfrak{M}$. □

Proof of Theorem 4.3.6. It remains to compare the deformation theories of the two moduli spaces to show that the homeomorphism $\mathfrak{M} \to \mathfrak{M}^{\text{hol}}$ is an isomorphism of real analytic spaces.

Step 3. *$\mathfrak{M}$ is isomorphic to the moduli space $\mathfrak{M}^\Sigma$ of solutions of* (4.3.1).

Let $\mathfrak{M}^\Sigma$ be the space of $\mathscr{G}(\Sigma, L)$–orbits of triples

$$(A, \alpha, \beta) \in \mathscr{A}(\Sigma, L) \times \Gamma(\Sigma, E^* \otimes L \otimes K^{1/2}) \times \Gamma(\Sigma, E \otimes L^* \otimes K^{1/2})$$

satisfying $(\alpha, \beta) \neq 0$ and

$$\begin{cases} \bar\partial_{AB}\alpha = 0, \\ \bar\partial_{AB}\beta = 0, \\ \alpha\beta = 0, \\ i * F_A + |\alpha|^2 - |\beta|^2 - i * \eta = 0 \end{cases}$$

Endow $\mathfrak{M}^\Sigma$ with a real analytic structure using local Kuranishi models. Let $\mathscr{C}_\Sigma^*$ and $\mathscr{C}^*$ be the spaces of irreducible configurations (A, Ψ) over Σ and M respectively, and $\mathscr{B}_\Sigma^*$, $\mathscr{B}_\Sigma$ their quotients by gauge groups. It follows from Proposition 4.3.2 and Proposition 4.1.6 that the inclusion

$$\mathcal{B}_\Sigma^* = \mathscr{C}_\Sigma^* / \mathscr{G}(\Sigma, L) \hookrightarrow \mathscr{C}^* / \mathscr{G}(\Sigma, L) = \mathcal{B}^*$$

induces a homeomorphism $\mathfrak{M}^\Sigma \to \mathfrak{M}$. The Seiberg–Witten moduli space $\mathfrak{M}$ is, at least locally, given as the zero set of a Fredholm section s of a bundle over $\mathcal{B}^*$. On the other hand, the restriction of s to $\mathcal{B}^*_\Sigma$ gives a Fredholm section defining $\mathfrak{M}^\Sigma$. Let $H^1_{A,\Psi} = \ker L_{A,\Psi}$ and $H^2_{A,\Psi} = \operatorname{coker} L_{A,\Psi}$ be the deformation and obstruction groups introduced in Section 2.3 and Section 3.2. Here, $L_{A,\Psi}$ is the self-adjoint elliptic operator governing the deformation theory of the Seiberg–Witten equation with two spinors; it can be identified with the derivative of s.

In order to show that the induced real analytic structures on $\mathfrak{M}$ and $\mathfrak{M}^\Sigma$ agree, we need to prove

$$H^1_{A,\Psi} = \ker \mathrm{d}s(A,\Psi) = \ker \mathrm{d}(s_{|\mathcal{B}^*_\Sigma})(A,\Psi) \tag{4.3.5}$$

for every $[A,\Psi] \in \mathfrak{M}^\Sigma = \mathfrak{M}$. The corresponding equality of cokernels follows then from the natural isomorphism between $H^1_{A,\Psi}$ and $H^2_{A,\Psi}$ (and likewise for the equations over Σ).

Equality (4.3.5) is the linearized version of Theorem 4.1.12. Let (A,Ψ) be an S^1–invariant solution. By Remark 3.2.8 and Proposition 4.1.8, $H^1_{A,\Psi}$ is the space of pairs

$$(a(t) + b(t)\mathrm{d}t, \phi(t)) \in \Gamma(S^1 \times \Sigma, \Lambda^1(i\mathbf{R}) \oplus \Lambda^0(i\mathbf{R}) \oplus (E^* \otimes S \otimes L))$$

satisfying

$$\begin{cases} i\left(\frac{\partial\phi}{\partial t} + b\Psi\right) + \sqrt{2}\sigma\left(\partial_{AB}\phi + a^{1,0}\Psi\right) = 0, \\ \frac{\partial a^{1,0}}{\partial t} + \partial b - i\mu_{\mathbf{C}}(\Psi,\phi) = 0, \\ *da + 2\mu_{\mathbf{R}}(\Psi,\phi) = 0, \\ -d^*a - \frac{\partial b}{\partial t} + i\,\mathrm{Im}\langle\Psi,\phi\rangle = 0. \end{cases}$$

Equality (4.3.5) will be established by showing that any solution $(a + b\mathrm{d}t, \phi)$ satisfies

$$\frac{\partial a}{\partial t} = 0, \quad \frac{\partial\phi}{\partial t} = 0, \quad b = 0.$$

This is done in the same way as in the proof of Theorem 4.1.12. First, apply $\partial/\partial t$ to the first two equations, then get rid of the terms $\partial\phi/\partial t$, $\partial b/\partial t$, and $\partial a^{1,0}/\partial t$. This results in

$$-\frac{\partial^2\phi}{\partial t^2} + 2\partial^*_{AB}\partial_{AB}\phi - i\,\mathrm{Im}\langle\Psi,\phi\rangle\Psi + d^*a\cdot\Psi + 2(\partial^*a^{1,0})\Psi + \sqrt{2}\sigma\mu_{\mathbf{C}}(\Psi,\phi)\Psi = 0,$$

$$-\frac{\partial^2 a^{1,0}}{\partial t^2} + \partial\partial^*a^{1,0} + i\partial\,\mathrm{Im}\langle\Psi,\phi\rangle + \sqrt{2}\mu_{\mathbf{C}}(\Psi,\sigma\partial\phi) + \sqrt{2}\mu_{\mathbf{C}}(\Psi,\sigma a^{1,0}\Psi) = 0.$$

Take the real L^2–product of the first equation with ϕ and the second equation with $a^{1,0}$. Integrating by parts as in the proof of Theorem 4.1.12, we obtain

$$\left\|\frac{\partial\phi}{\partial t}\right\|^2_{L^2} + \left\|\frac{\partial a}{\partial t}\right\|^2_{L^2} + 2\left\|\partial_{AB}\phi + a^{1,0}\Psi\right\|^2_{L^2} + \|-d^*a + i\,\mathrm{Im}\langle\Psi,\phi\rangle\|^2_{L^2} + \sqrt{2}\,\|\mu_{\mathbf{C}}(\Psi,\phi)\|^2_{L^2} = 0.$$

We have used identity (4.1.2) to relate $\mu_{\mathbf{C}}$ to the inner product. Thus, we have proved that $b = 0$, ϕ and a are pulled back from Σ and satisfy

$$\begin{cases} \partial_{AB}\phi + a^{1,0}\Psi = 0, \\ \mu_{\mathbf{C}}(\Psi,\phi) = 0, \\ *da + 2\mu_{\mathbf{R}}(\Psi,\phi) = 0, \\ -d^*a - i\,\mathrm{Im}\langle\Psi,\phi\rangle = 0. \end{cases} \tag{4.3.6}$$

Recall that we identify $\overline{\Psi}$ with a pair (α, β). After a conjugation equation (4.3.6) translates to the following equation for a and $\overline{\phi} = (u, v)$

$$\begin{cases} \bar{\partial}_{AB} u + a^{0,1}\alpha = 0, \\ \bar{\partial}_{AB} v - a^{0,1}\beta = 0, \\ \alpha v + u\beta = 0, \\ *ida + 2\operatorname{Re}\langle \alpha, u\rangle - 2\operatorname{Re}\langle \beta, v\rangle = 0, \\ -d^* a - i\operatorname{Im}\langle \alpha, u\rangle - i\operatorname{Im}\langle \beta, v\rangle = 0. \end{cases} \tag{4.3.7}$$

This is the linearization of (4.3.1) together with the Coulomb gauge fixing condition. We conclude that (4.3.5) holds and $\mathfrak{M}$ is isomorphic to $\mathfrak{M}^{\Sigma}$ as real analytic spaces.

Step 4. *$\mathfrak{M}^{\Sigma}$ is isomorphic to $\mathfrak{M}^{\mathrm{hol}}$.*

The proof is similar to that of [FM94, Theorem 2.6]. As before, the main point is to show an isomorphism of the deformation spaces for $\mathfrak{M}^{\Sigma}$ and $\mathfrak{M}^{\mathrm{hol}}$. The former is given by (4.3.7) and the latter consists of solutions of the first three equations together with a choice of a local slice for the action of $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$. The Lie algebra of $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$ splits is the direct sum of $C^\infty(\Sigma, \mathbf{R})$ and the Lie algebra of $\mathscr{G}(\Sigma, L)$. Under this splitting, we can choose a slice of $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$–action imposing the standard Coulomb gauge condition for $\mathscr{G}(\Sigma, L)$, which is the last equation of (4.3.7), together with a choice of a slice for the action of $C^\infty(\Sigma, \mathbf{R})$:

$$e^f(A, \alpha, \beta) = (A + \partial f - \bar{\partial} f, e^f\alpha, e^{-f}\beta).$$

The linearization of this action at (A, α, β) is

$$f \mapsto (-\bar{\partial} f + \partial f, f\alpha, -f\beta). \tag{4.3.8}$$

A local slice for the action of $C^\infty(\Sigma, \mathbf{R})$ can be obtained from any subspace of

$$\Omega^1(\Sigma, i\mathbf{R}) \oplus \Gamma(E^* \otimes L \otimes K^{1/2}) \oplus \Gamma(E \otimes L^* \otimes K^{1/2})$$

which is complementary to the image of (4.3.8). Hence, to show that the deformation spaces of $\mathfrak{M}^{\Sigma}$ and $\mathfrak{M}^{\mathrm{hol}}$ are isomorphic it is enough to prove that the subspace given by

$$i * da + 2\operatorname{Re}\langle \alpha, u\rangle - 2\operatorname{Re}\langle \beta, v\rangle = 0$$

is complementary to the image of (4.3.8). In other words, we need to know that for any triple (a, u, v) there is a unique function $f \in C^\infty(\Sigma, \mathbf{R})$ such that

$$\begin{aligned} 0 &= i * d(a - \bar{\partial} f + \partial f) + 2\operatorname{Re}\langle \alpha, u + f\alpha\rangle - 2\operatorname{Re}\langle \beta, v - f\beta\rangle \\ &= \left\{ \Delta + 2(|\alpha|^2 + |\beta|^2) \right\} f + i * da + 2\operatorname{Re}\langle \alpha, u\rangle - 2\operatorname{Re}\langle \beta, v\rangle. \end{aligned}$$

This is true because $(\alpha, \beta) \neq 0$ and so the operator $\Delta + 2(|\alpha|^2 + |\beta|^2)$ is invertible on $C^\infty(\Sigma, \mathbf{R})$. In the same way as in [FM94, Theorem 2.6] we conclude that (4.3.7) provides a local Fredholm model for both $\mathfrak{M}^{\Sigma}$ and $\mathfrak{M}^{\mathrm{hol}}$ and so the two spaces have isomorphic analytic structures. □

4.4 A TALE OF TWO COMPACTIFICATIONS

The goal of this section is to define two natural compactifications $\mathfrak{M} \subset \overline{\mathfrak{M}}$ and $\mathfrak{M}^{\mathrm{hol}} \subset \overline{\mathfrak{M}}^{\mathrm{hol}}$ using, respectively, gauge theory and complex geometry. We then extend the isomorphism $\mathfrak{M} \cong \mathfrak{M}^{\mathrm{hol}}$ to a homeomorphism between these compactifications.

Theorem 4.4.1. *The isomorphism of real analytic spaces $\mathfrak{M} \cong \mathfrak{M}^{\mathrm{hol}}$ extends to a homeomorphism $\overline{\mathfrak{M}} \cong \overline{\mathfrak{M}}^{\mathrm{hol}}$.*

We assume $d - \tau < 0$ so in particular there are no reducible solutions. The discussion can be easily adapted to the cases $d - \tau = 0$ and $d - \tau > 0$.

4.4.1 *A complex-geometric compactification*

$\mathfrak{M}^{\mathrm{hol}}$ has a natural compactification analogous to the one described in [BW96]. Consider the subspace $S \subset \mathscr{C}_\Sigma \times \mathbf{C}$ given by

$$S := \{(A, \alpha, \beta, t) \mid (A, \alpha, \beta) \text{ satisfies equations (4.3.2)}, \alpha \neq 0, \text{ and } (\beta, t) \neq (0,0)\}.$$

The group $\mathscr{G}^{\mathbf{C}}(\Sigma, L) \times \mathbf{C}^*$ acts freely on S by the standard action of the first factor on $\mathscr{C}_\Sigma$ and

$$\lambda(A, \alpha, \beta, t) = (A, \alpha, \lambda\beta, \lambda t) \quad \text{for } \lambda \in \mathbb{C}^\times.$$

Definition 4.4.2. We define $\overline{\mathfrak{M}}^{\mathrm{hol}}$ to be the quotient of S by $\mathscr{G}^{\mathbf{C}}(\Sigma, L) \times \mathbf{C}^*$.

This is analogous to compactifying $\mathbf{C}^N$ by $\mathbb{CP}^N$ which is the quotient of $(\mathbf{C}^N \times \mathbf{C}) \backslash \{(0,0)\}$ by the free action of $\mathbf{C}^*$; in fact, $\overline{\mathfrak{M}}^{\mathrm{hol}}$ is obtained by applying this construction fiberwise.

Definition 4.4.3. Let $\mathfrak{N}$ be the subspace of $\mathfrak{M}^{\mathrm{hol}}$ consisting of triples of the form $(A, \alpha, 0)$. Equivalently, $\mathfrak{N}$ is the space of $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$–orbits of pairs (A, α) satisfying $\bar{\partial}_{AB}\alpha = 0$ and $\alpha \neq 0$.

(In this section we fix the degree d of L and parameters $(\mathbf{p}, \eta)$, but keep in mind that, in general $\mathfrak{N} = \mathfrak{N}_d(\mathbf{p}, \eta)$ depends on the choice of this data, just like $\mathfrak{M}^{\mathrm{hol}} = \mathfrak{M}^{\mathrm{hol}}_d(\mathbf{p}, \eta)$ and $\mathfrak{M} = \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$.)

We will see momentarily that $\mathfrak{N}$ is compact. The natural projection $(A, \alpha, \beta) \mapsto (A, \alpha)$ induces a surjective map $\pi \colon \mathfrak{M}^{\mathrm{hol}} \to \mathfrak{N}$. Let $[A, \alpha] \in \mathfrak{N}$ and denote by $\mathscr{L}_A$ the holomorphic structure on L induced by A. The fiber $\pi^{-1}([A, \alpha])$ is the kernel of the homomorphism

$$H^0(\Sigma, \mathscr{E}^* \otimes \mathscr{L}_A \otimes K^{1/2}) \xrightarrow{\alpha} H^0(\Sigma, K)$$

given by pairing with α. The compactification $\overline{\mathfrak{M}}^{\mathrm{hol}}$ is obtained by replacing each fiber $\ker \alpha$ with the projective space $\mathbb{P}(\ker \alpha \oplus \mathbf{C})$ containing it.

Proposition 4.4.4. *The space $\overline{\mathfrak{M}}^{\mathrm{hol}}$ is metrizable, compact, and contains $\mathfrak{M}^{\mathrm{hol}}$ as an open dense subset. Moreover, the complex analytic structure on $\mathfrak{M}^{\mathrm{hol}}$ extends to a complex analytic structure on $\overline{\mathfrak{M}}^{\mathrm{hol}}$ with respect to which $\mathfrak{M}^{\mathrm{hol}}$ is Zariski open.*

Proof. It is clear that $\overline{\mathfrak{M}}^{\text{hol}}$ is metrizable and $\mathfrak{M}^{\text{hol}} \subset \overline{\mathfrak{M}}^{\text{hol}}$ is open and dense. In order to show that $\overline{\mathfrak{M}}^{\text{hol}}$ is compact, consider a sequence $[A_i, \alpha_i, \beta_i, t_i] \in \overline{\mathfrak{M}}^{\text{hol}}$; we need to argue that there are sequences $h_i \in \mathscr{G}^{\mathbf{C}}(\Sigma, L)$ and $\lambda_i \in \mathbf{C}^*$ such that after passing to a subsequence $h_i \lambda_i (A_i, \alpha_i, \beta_i, t_i)$ converges smoothly in S. This is the content of Step 1 in [Doa17a, Proof of Theorem 2.2].

Let $\mathscr{C}^*_\Sigma \subset \mathscr{C}_\Sigma$ be the subset of configurations (A, α, β) with $\alpha \neq 0$. The group $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$ acts freely on $\mathscr{C}^*_\Sigma$ with quotient $\mathcal{B}^*_\Sigma$, a complex Banach manifold. There is a holomorphic vector bundle $\mathcal{W} \to \mathcal{B}^*_\Sigma$ such that $\mathfrak{M}^{\text{hol}}$ is the zero set of a holomorphic Fredholm section $\mathcal{S} \colon \mathcal{B}^*_\Sigma \to \mathcal{W}$. The zero set of such a section carries a natural complex analytic structure [FM94, Sections 4.1.3-4.1.4]. The complex analytic structure on $\mathfrak{M}^{\text{hol}}$ is extended to $\overline{\mathfrak{M}}^{\text{hol}}$ by extending $\mathcal{S}$ to a Fredholm section whose zero set is $\overline{\mathfrak{M}}^{\text{hol}}$. Replace $\mathscr{C}^*_\Sigma$ by the subspace of $\mathscr{C}_\Sigma \times \mathbf{C}$ consisting of quadruples (A, α, β, t) for which $\alpha \neq 0$ and $(\beta, t) \neq (0, 0)$. Let $\overline{\mathcal{B}}^*_\Sigma$ be the quotient of this space by the action of $\mathscr{G}^{\mathbf{C}}(\Sigma, L) \times \mathbf{C}^*$: it contains $\mathcal{B}^*_\Sigma$ as an open subset. Let $\overline{\mathcal{W}} \to \overline{\mathcal{B}}^*_\Sigma$ be the vector bundle obtained as the quotient of $\mathcal{W} \times \mathbf{C}^*$ by the lifted action of $\mathscr{G}^{\mathbf{C}}(\Sigma, L) \times \mathbf{C}^*$. There is a holomorphic Fredholm section $\overline{\mathcal{S}}$ extending $\mathcal{S}$ so that $\overline{\mathfrak{M}}^{\text{hol}} = \overline{\mathcal{S}}^{-1}(0)$. When restricted to the open subset $\mathcal{B}^*_\Sigma$, this reduces to the construction of $\mathfrak{M}^{\text{hol}}$ described above, so the inclusion $\mathfrak{M}^{\text{hol}} \subset \overline{\mathfrak{M}}^{\text{hol}}$ is compatible with the induced analytic structures. Moreover, $\overline{\mathfrak{M}}^{\text{hol}} \backslash \mathfrak{M}^{\text{hol}}$ is the intersection of $\overline{\mathcal{S}}^{-1}(0)$ with the analytic subset $\overline{\mathcal{B}}^*_\Sigma \backslash \mathcal{B}^*_\Sigma$ given by the equation $t = 0$. We conclude that $\overline{\mathfrak{M}}^{\text{hol}} \backslash \mathfrak{M}^{\text{hol}}$ is an analytic subset of $\overline{\mathfrak{M}}^{\text{hol}}$, and so $\mathfrak{M}^{\text{hol}}$ is Zariski open. □

Corollary 4.4.5. *The subspace $\mathfrak{N} \subset \mathfrak{M}^{\text{hol}}$ is compact. Furthermore, $\mathfrak{M}^{\text{hol}}$ is compact if and only if $\mathfrak{N} = \mathfrak{M}^{\text{hol}}$.*

Proof. $\mathfrak{N}$ consists of equivalence classes $[A, \alpha, \beta]$ for which $\beta = 0$; it is compact by Step 1 in [Doa17a, Proof of Theorem 1.3]. If $\mathfrak{N} = \mathfrak{M}^{\text{hol}}$, then $\mathfrak{M}^{\text{hol}}$ is compact. To prove the converse statement, observe that if $\mathfrak{M}^{\text{hol}}$ is non-compact, then $\overline{\mathfrak{M}}^{\text{hol}} \backslash \mathfrak{M}^{\text{hol}}$ is non-empty by Proposition 4.4.4. On the other hand, $\overline{\mathfrak{M}}^{\text{hol}} \backslash \mathfrak{M}^{\text{hol}}$ consists of $\mathscr{G}^{\mathbf{C}}(\Sigma, L) \times \mathbf{C}^*$–orbits of the form $[A, \alpha, \beta, 0]$ with $\beta \neq 0$ so every element of $\overline{\mathfrak{M}}^{\text{hol}} \backslash \mathfrak{M}^{\text{hol}}$ gives rise to an element of $\mathfrak{M}^{\text{hol}} \backslash \mathfrak{N}$. □

4.4.2 *A gauge-theoretic compactification*

For an arbitrary 3–manifold a good compactification of $\mathfrak{M}$ is yet to be constructed—see [Doa17a, Introduction] for a discussion of analytical difficulties involved in such a construction. However, for $M = S^1 \times \Sigma$ we can overcome these obstacles thanks to a refined compactness theorem.

Theorem 4.4.6 ([Doa17a])**.** *If $(A_i, \Psi_i = (\alpha_i, \bar{\beta}_i))$ is a sequence of solutions of (4.3.1) with $\|\Psi_i\|_{L^2} \to \infty$, then after passing to a subsequence and applying gauge transformations $(A_i, \Psi_i / \|\Psi_i\|_{L^2})$ converges in C^∞_{loc} on the complement of a finite set $D = \{x_1, \ldots, x_N\}$. The limiting configuration $(A, \Psi = (\alpha, \bar{\beta}))$ is defined on $\Sigma \backslash D$ and satisfies*

- $\|\Psi\|_{L^2} = 1$ *and* $|\Psi| > 0$ *on* $\Sigma \backslash D$,
- $\bar{\partial}_{AB} \alpha = 0$, $\bar{\partial}_{AB} \beta = 0$, $\alpha\beta = 0$, *and* $|\alpha| = |\beta|$ *on* $\Sigma \backslash D$.

- *A is flat on $\Sigma\backslash D$ and has holonomy contained in $\mathbf{Z}_2$.*
- *There are non-zero integers $a_1, \ldots, a_N$ such that $\sum_{k=1}^N a_k = 2d$ and*

$$*\frac{i}{2\pi} F_{A_i} \longrightarrow \frac{1}{2} \sum_{k=1}^{N} a_k \delta_{x_k}$$

  in the sense of measures.

- *For each $k = 1, \ldots, N$ we have*

$$|\Psi(x)| = O(\mathrm{dist}(x_k, x)^{|a_k|/2}).$$

Definition 4.4.7. Let $D \subset \Sigma$ be a finite set. We say that a gauge transformation in $\mathscr{G}(\Sigma\backslash D)$ or $\mathscr{G}^c(\Sigma\backslash D)$ is *simple* if it has degree zero around each point of D. Denote by $\mathscr{G}_0(\Sigma\backslash D)$ and $\mathscr{G}_0^c(\Sigma\backslash D)$ the subgroups of simple gauge transformations.

Definition 4.4.8. Let $D \subset \Sigma$ be a finite subset. With any flat connection $A \in \mathscr{A}(\Sigma\backslash D, L)$ we associate a measure $i * F_A$ on Σ as follows. For $x \in D$ let $B \subset \Sigma$ be a small disc centred at x and not containing other points of D. In a unitary trivialisation of $L_{|B}$ we have $A = d + a$ for a one-form $a \in \Omega^1(B\backslash\{x\}, i\mathbf{R})$. Denote

$$q_x = \int_{\partial B} ia.$$

The measure $i * F_A$ is defined by

$$i * F_A := \sum_{x \in D} q_x \delta_x.$$

One easily checks that $i * F_A$ is well-defined and invariant under simple gauge equivalences.

Definition 4.4.9. A *limiting configuration* is a triple (A, Ψ, D) comprising of a finite subset $D = \{x_1, \ldots, x_N\} \subset \Sigma$, a connection $A \in \mathscr{A}(\Sigma\backslash D, L)$, and a pair $\Psi = (\alpha, \bar{\beta})$ of nowhere-vanishing sections $\alpha \in \Gamma(\Sigma\backslash D, E^* \otimes L \otimes K^{1/2})$ and $\beta \in \Gamma(\Sigma\backslash D, E \otimes L^* \otimes K^{1/2})$ satisfying

- $\|\Psi\|_{L^2} = 1$ and $|\Psi| > 0$ on $\Sigma\backslash D$
- $\bar{\partial}_{AB}\alpha = 0$, $\bar{\partial}_{AB}\beta = 0$, $\alpha\beta = 0$, and $|\alpha| = |\beta|$ on $\Sigma\backslash D$.
- A is flat on $\Sigma\backslash D$ and has holonomy contained in $\mathbf{Z}_2$.
- There are non-zero integers $a_1, \ldots, a_N$ such that $\sum_{k=1}^N a_k = 2d$ and

$$*\frac{i}{2\pi} F_A = \frac{1}{2} \sum_{k=1}^{N} a_k \delta_{x_k} \quad \text{as measures.}$$

- For each $k = 1, \ldots, N$ we have

$$|\Psi(x)| = O(\mathrm{dist}(x_k, x)^{|a_k|/2}).$$

(A, Ψ, D) and (A', Ψ', D') are *simple gauge equivalent* if $D = D'$ and they differ by an element $u \in \mathscr{G}_0(\Sigma\backslash D)$.

Let $\mathcal{F}$ be the set of simple gauge equivalence classes of limiting configurations.

The space $\mathcal{F}$ can be equipped with a natural topology in which a sequence $[A_i, \Psi_i, D_i]$ converges to $[A, \Psi, D]$ if and only if $i * F_{A_i'} \to i * F_A$ weakly as measures and after applying a sequence in $\mathscr{G}_0(\Sigma \backslash D)$ we have $A_i \to A$ and $\Psi_i \to \Psi$ in C^∞_{loc} on $\Sigma \backslash D$.

Definition 4.4.10. Let $[A, \Psi, D]$ be an equivalence class in $\mathcal{F}$. For $\varepsilon > 0$, $\delta > 0$, an integer $k \geqslant 0$, and a continuous function $f \colon \Sigma \to \mathbf{R}$ we define $V_{\varepsilon,\delta,k,f}(A, \Psi, D) \subset \mathcal{F}$ to be the set of the elements of $\mathcal{F}$ which have a representative (A', Ψ', D') satisfying

- $D' \subset D_\varepsilon$ where $D_\varepsilon := \{x \in \Sigma \mid \mathrm{dist}(x, D) < \varepsilon\}$,
- $\|A' - A\|_{C^k(\Sigma \backslash D_\varepsilon)} < \delta$,
- $\|\Psi' - \Psi\|_{C^k(\Sigma \backslash D_\varepsilon)} < \delta$.
- $\left| \int_\Sigma (i * F_{A'}) f - \int_\Sigma (i * F_A) f \right| < \delta$.

Lemma 4.4.11. *The family of subsets*

$$\left\{ V_{\varepsilon,\delta,k,f}(A, \Psi, D) \right\}$$

forms a base of a Hausdorff topology on $\mathcal{F}$.

The proof is a simple application of [DK90, Proposition 2.3.15]. The next step is to combine $\mathfrak{M}$ and $\mathcal{F}$ into one topological space. For this purpose it is convenient to identify points of $\mathfrak{M}$ with gauge equivalence classes of triples (A, Ψ, t), where $t \in (0, \infty)$, $\|\Psi\|_{L^2} = 1$, and

$$\begin{cases} \not{D}_{AB} \Psi = 0, \\ t^2 F_A = \mu(\Psi). \end{cases} \tag{4.4.1}$$

The map $(A, \Psi, t) \mapsto (A, t^{-1}\Psi)$ gives a homeomorphism between the space of such classes and the moduli space $\mathfrak{M}$. Recall that in our setting there are no reducibles so there is no boundary at $t \to \infty$. The boundary at $t \to 0$ is obtained by attaching the space of limiting configurations.

Definition 4.4.12. Let $[A, \Psi, D]$ be an equivalence class in $\mathcal{F}$. For $\varepsilon > 0$, $\delta > 0$, an integer $k \geqslant 0$, and a continuous function $f \colon \Sigma \to \mathbf{R}$ we define

$$W_{\varepsilon,\delta,k,f}(A, \Psi, D) \subset \mathfrak{M}$$

to be the set of the elements of $\mathfrak{M}$ that have a representative (A', Ψ', t) satisfying

- $t < \delta$,
- $\|A' - A\|_{C^k(\Sigma \backslash D_\varepsilon)} < \delta$,
- $\|\Psi' - \Psi\|_{C^k(\Sigma \backslash D_\varepsilon)} < \delta$,
- $\left| \int_\Sigma (i * F_{A'}) f - \int_\Sigma (i * F_A) f \right| < \delta$.

Definition 4.4.13. The *compactified moduli space* is

$$\overline{\mathfrak{M}} := \mathfrak{M} \cup \mathcal{F}$$

equipped with the topology whose basis consists of the subsets

$$\overline{W}_{\varepsilon,\delta,k,f}(A,\Psi,D) := W_{\varepsilon,\delta,k,f}(A,\Psi,D) \cup V_{\varepsilon,\delta,k,f}(A,\Psi,D),$$

for all choices of $\varepsilon, \delta, k, f$.

Lemma 4.4.14. *Let $\mathcal{U}$ be a base of the topology on $\mathfrak{M}$. The family of subsets*

$$\{\overline{W}_{\varepsilon,\delta,k}(A,\Psi,D)\} \cup \mathcal{U}$$

forms a base of a Hausdorff topology on $\overline{\mathfrak{M}}$.

4.4.3 *A homeomorphism at infinity*

The main ingredient in the proof of Theorem 4.4.1 is

Proposition 4.4.15. *The spaces $\overline{\mathfrak{M}}^{\mathrm{hol}} \backslash \mathfrak{M}^{\mathrm{hol}}$ and $\mathcal{F}$ are homeomorphic.*

The proof of Proposition 4.4.15 is preceded by auxiliary results about limiting configurations. The first of them is a complex-geometric analogue of the statement that a limiting configuration induces a flat connection with $\mathbf{Z}_2$ holonomy.

Lemma 4.4.16. *Let (A, α, β) be a solution of (4.3.2) with $\alpha \neq 0$ and $\beta \neq 0$. Denote by $\mathscr{L}$ the holomorphic line bundle $(L, \bar{\partial}_A)$. Let D_1 and D_2 be the zero divisors of α and β respectively, and $\mathcal{O}(D_1 - D_2)$ the holomorphic line bundle associated to $D_1 - D_2$. There is a canonical holomorphic isomorphism*

$$\varphi_{\alpha\beta} \colon \mathcal{O}(D_1 - D_2) \longrightarrow \mathscr{L}^2.$$

Proof. Recall that α is a holomorphic section of $\mathscr{E} \otimes \mathscr{L} \otimes K^{1/2}$ and β is a holomorphic section of $\mathscr{E}^* \otimes \mathscr{L}^* \otimes K^{1/2}$. Since $\alpha\beta = 0$ and the rank of $\mathscr{E}$ is two, we have the short exact sequence

$$0 \longrightarrow \mathscr{L}^{-1} \otimes K^{-1/2} \otimes \mathcal{O}(D_1) \xrightarrow{\ \alpha\ } \mathscr{E} \xrightarrow{\ \beta\ } \mathscr{L}^{-1} \otimes K^{1/2} \otimes \mathcal{O}(-D_2) \longrightarrow 0. \tag{4.4.2}$$

The associated isomorphism of the determinant line bundles is

$$\mathscr{L}^{-2} \otimes \mathcal{O}(D_1 - D_2) \cong \det \mathscr{E} \cong \mathcal{O},$$

where the last isomorphism follows from the fact that $\mathscr{E}$ is a holomorphic $SL(2,\mathbf{C})$–bundle. Tensoring both sides with $\mathscr{L}^2$, we obtain the desired isomorphism $\varphi_{\alpha\beta}$. It is canonically determined by α and β. □

The lemma below provides an upper bound on the number of components, counted with multiplicities, of the singular set of a limiting configuration.

Lemma 4.4.17. *There exists $M \geqslant 0$, depending only on the holomorphic bundle $\mathscr{E}$, with the following significance. If $(\mathscr{L}, \alpha, \beta)$ is a triple as in Lemma 4.4.16, then*

$$\deg D_1 + \deg D_2 \leqslant M.$$

Proof. Tensoring exact sequence (4.4.2) with $K^{1/2}$ we see that $\mathscr{L}^{-1} \otimes \mathcal{O}(D_1)$ is a holomorphic subbundle of $\mathscr{E} \otimes K^{1/2}$. It is an elementary fact that the degrees of line subbundles of a given holomorphic bundle are bounded above [Muk03, Corollary 10.9]. In fact, if $\mathcal{F}$ is a holomorphic vector bundle and $A \subset \mathcal{F}$ a line subbundle, then

$$\deg A \leqslant h^0(\Sigma, \mathcal{F}) + 2g(\Sigma) - 2 \tag{4.4.3}$$

where $g(\Sigma)$ is the genus of Σ. (We will use this bound later.) Thus, we have an upper bound on the degree of $\mathscr{L}^{-1} \otimes \mathcal{O}(D_1)$. On the other hand, $\mathscr{L}^2$ is isomorphic to $\mathcal{O}(D_1 - D_2)$, so

$$\deg\left(\mathscr{L}^{-1} \otimes \mathcal{O}(D_1)\right) = -\deg \mathscr{L} + \deg D_1 = \frac{1}{2}\left(\deg D_1 + \deg D_2\right),$$

which proves the lemma. □

The next result will be useful in proving the convergence of measures.

Lemma 4.4.18. *Let $f\colon \Sigma \to \mathbf{R}$ be a continuous function, $\gamma > 0$, and $D \subset \Sigma$ a finite subset. Then there exist $\varepsilon > 0$ and $\delta > 0$ with the following property. Suppose that $D' \subset \Sigma$ is another finite subset, and A and A' are two flat connections over $\Sigma \backslash D$ and $\Sigma \backslash D'$ respectively, satisfying*

- $D' \subset D_\varepsilon$,
- $\|A' - A\|_{C^0(\Sigma \backslash D_\varepsilon)} < \delta$,
- *the measures $*iF_A$ and $*iF_{A'}$ have integer weights.*

Then we have

$$\left| \int_\Sigma (i * F_{A'}) f - \int_\Sigma (i * F_A) f \right| \leqslant \gamma \|i * F_{A'}\|.$$

where $\|i * F_{A'}\|$ is the total variation *of the measure $i * F_{A'}$ given by*

$$\|i * F_{A'}\| = \sum_{x \in D'} |q_x|$$

for $i * F_{A'} = \sum_{x \in D'} q_x \delta_x$.

Proof. Let $D = \{x_1, \dots, x_N\}$ and $a_1, \dots, a_N$ be the integer weights of the measure $i * F_A$ as in Definition 4.4.8. Suppose that ε is small enough so that the discs B_i of radius ε centred x_i are pairwise disjoint. Partition the set D' into disjoint subsets $D'_1, \dots, D'_N$ consisting of points within ε–distance from $x_1, \dots, x_N$ respectively. For each i choose small disjoint discs $E_{i1}, E_{i2}, \dots$ centred at points of D'_i and contained in B_i. Let $b_{i1}, b_{i2} \dots$ be the weights of the points in D'_i in the measure $i * F_{A'}$.

In a unitary trivialisation of L over each B_i we have

$$A = d + a \qquad \text{and} \qquad A' = d + a',$$

where one-form a is defined on $B_i \backslash \{x_i\}$ and a' is defined on $B_i \backslash D'_i$. By the hypothesis of the lemma $\|a - a'\|_{C^0(\partial B_i)} < \delta$. Thus, for sufficiently small δ,

$$\left| a_i - \sum_j b_{ij} \right| = \left| \int_{\partial B_i} ia - \sum_j \int_{\partial E_{ij}} ia' \right| = \left| \int_{\partial B_i} ia - \int_{\partial B_i} ia' \right| < 1.$$

Since all numbers a_i, b_{ij} are integers, so we conclude that

$$a_i = \sum_j b_{ij}.$$

For each i denote the points of D_i' by $\{x_{ij}\}$. Then

$$\begin{aligned} \left| \int_\Sigma (i * F_{A'}) f - \int_\Sigma (i * F_A) f \right| &= \left| \sum_i a_i f(x_i) - \sum_{ij} b_{ij} f(x_{ij}) \right| \\ &\leqslant \sum_{ij} |b_{ij}| \left| f(x_i) - f(x_{ij}) \right| . \end{aligned}$$

By the continuity of f we can choose ε sufficiently small so that

$$\sup_{x \in B_i} |f(x_i) - f(x)| \leqslant \gamma$$

for all $i = 1, \ldots, N$. Then

$$\left| \int_\Sigma (i * F_{A'}) f - \int_\Sigma (i * F_A) f \right| \leqslant \gamma \| i * F_{A'}. \| \qquad \square$$

The last lemma allows us to extend a limiting configuration to a holomorphic section. The proof is a minor variation of [Doa17a, Proof of Lemma 2.1].

Lemma 4.4.19. *Let L be a Hermitian line bundle over the unit disc $B \subset \mathbf{C}$. Suppose that A is a unitary connection on $L_{|B \backslash \{0\}}$ and φ a section of L over $B \backslash \{0\}$ satisfying $\bar{\partial}_A \varphi = 0$ and $|\varphi| = 1$. Denote by $\deg \varphi$ the degree of $\varphi_{|\partial B}$. Then*

1. *$F_A = 0$ on $B \backslash \{0\}$ and $i * F_A = (2\pi \deg \varphi) \delta_0$ as measures.*
2. *There exists a complex gauge transformation $h \colon B \backslash \{0\} \to \mathbf{C}^*$ such that h has degree zero around zero and in some trivialisation of L around zero $h(A)$ is the trivial connection and $h\varphi = z^k$.*

Proof of Proposition 4.4.15. Set $\mathcal{X} = \overline{\mathfrak{M}}^{\mathrm{hol}} \backslash \mathfrak{M}^{\mathrm{hol}}$. We will construct a continuous bijection $\mathcal{X} \to \mathcal{F}$. Since the domain is compact by Proposition 4.4.4 and the target space is Hausdorff by Lemma 4.4.11, such a map is necessarily a homeomorphism.

Step 1. *The construction of $\mathcal{X} \to \mathcal{F}$.*

Let $[A, \alpha, \beta, t] \in \mathcal{X}$. By definition, $t = 0$, $\alpha \neq 0$, $\beta \neq 0$, and (4.3.2) is satisfied. Let D_1, D_2 be the zero divisors of α, β, respectively. We will interchangeably consider them as divisors or as subsets of Σ. Set $D = D_1 \cup D_2$. We claim that there is a simple complex gauge transformation $h \in \mathscr{G}_0^c(\Sigma \backslash D)$ such that $(hA, h\alpha, h^{-1}\beta)$ is a limiting configuration. A necessary condition is

$$|h\alpha| = |h^{-1}\beta| \quad \text{on } \Sigma \backslash D. \tag{4.4.4}$$

A transformation satisfying (4.4.4) exists since α and β are both non-zero on $\Sigma \backslash D$ and we can set $h = \sqrt{|\beta| / |\alpha|}$; any other choice of h will differ from that one by an element of $\mathscr{G}_0(\Sigma \backslash D)$.

The map $\mathcal{X} \to \mathcal{F}$ is defined by $[A, \alpha, \beta, 0] \mapsto (A', \alpha', \beta') := (h(A), h\alpha, h^{-1}\beta)$. We need to show that (A', α', β') represents a class in $\mathcal{F}$. We clearly have

$$\begin{cases} \bar{\partial}_{A'B}\alpha' = 0, \\ \bar{\partial}_{A'B}\beta' = 0, \\ |\alpha'| = |\beta'|. \end{cases}$$

Moreover, for $\Psi' := (\alpha', \beta')$

$$|\Psi'| = \sqrt{|\alpha'|^2 + |\beta'|^2} = \sqrt{|h|^2|\alpha|^2 + |h|^{-2}|\beta|^2} = \sqrt{2|\alpha||\beta|}.$$

Integrating over Σ yields

$$\|\Psi'\|_{L^2} = \sqrt{\int_\Sigma 2|\alpha||\beta|}.$$

After rescaling β, which does not change the class of $[A, \alpha, \beta, 0] \in \mathcal{X}$, we can assume $\|\Psi\|_{L^2} = 1$. We also see that $|\Psi| > 0$ on $\Sigma \setminus D$ and in a neighbourhood of every $x \in D$

$$|\Psi(y)| = O\left(\operatorname{dist}(y, x)^{\frac{\operatorname{ord}_x(\alpha) + \operatorname{ord}_x(\beta)}{2}}\right), \tag{4.4.5}$$

where $\operatorname{ord}_x(\alpha)$ and $\operatorname{ord}_x(\beta)$ denote the order of vanishing of α and β at x.

It remains to prove that A' is flat and the measure $i * F_{A'}$ is as in Definition 4.4.9. Let $\varphi_{\alpha\beta} \colon \mathcal{O}(D_1 - D_2) \to L^2$ be the A–holomorphic isomorphism from Lemma 4.4.16. The construction of Lemma 4.4.16 is local, so we can also define an analogous map $\varphi_{\alpha'\beta'}$ corresponding to sections α' and β'. Since they are both nowhere vanishing, $\varphi_{\alpha'\beta'}$ is an A'–holomorphic isomorphism of the trivial bundle over $\Sigma \setminus D$ and $L^2_{|\Sigma \setminus D}$; thus, $\varphi_{\alpha'\beta'}$ is a nowhere vanishing A'–holomorphic section of L^2 over $\Sigma \setminus D$. Moreover, on $\Sigma \setminus D$

$$|\varphi_{\alpha'\beta'}| = \frac{|\alpha'|}{|\beta'|} = 1 \quad \text{and} \quad \varphi_{\alpha'\beta'} = h^2 \varphi_{\alpha\beta}$$

By Lemma 4.4.19, the tensor product connection $A \otimes A$ on L^2 is flat; so A itself is flat and for every $x \in D$ the weight of the measure $i/2\pi * F_A$ at x is equal to half of the degree of $\varphi_{\alpha'\beta'}$ around x. Since h has degree zero around each point $x \in D$, the degrees of $\varphi_{\alpha\beta}$ and $\varphi_{\alpha'\beta'}$ around x agree. Denote this degree by $q_x \in \mathbf{Z}$. Since the zero divisor of $\varphi_{\alpha\beta}$ is $D_1 - D_2$,

$$\sum_{x \in D} q_x = \deg D_1 - \deg D_2 = \deg(L^2) = 2d.$$

Finally, observe that for every $x \in D$ we have

$$q_x = \operatorname{ord}_x(\alpha) - \operatorname{ord}_x(\beta) \leqslant \operatorname{ord}_x(\alpha) + \operatorname{ord}_x(\beta).$$

Together with (4.4.5) this shows that (A', α', β', D) is a limiting configuration. It is easy to check that if we replace $(A, \alpha, \beta, 0)$ by a different quadruple in the same orbit of the $\mathscr{G}^{\mathbf{C}}(\Sigma, L) \times \mathbf{C}^*$–action, then the resulting limiting configurations are simple gauge-equivalent.

Step 2. *$\mathcal{X} \to \mathcal{F}$ is injective.*

Suppose that $(A_1, \alpha_1, \beta_1, 0)$ and $(A_2, \alpha_2, \beta_2, 0)$ give rise to limiting configurations that are simple gauge equivalent. In particular, they have the same singular set, D say. Suppose that β_1 and β_2 are scaled so that

$$\int_\Sigma 2|\alpha_1||\beta_1| = \int_\Sigma 2|\alpha_2||\beta_2| = 1.$$

Composing the simple gauge equivalence of the limiting configurations with complex gauge transformations satisfying (4.4.4), we obtain $t \in \mathscr{G}_0^c(\Sigma\backslash D)$ such that on $\Sigma\backslash D$

$$A_2 = t(A_1), \quad \alpha_2 = t\alpha_1, \quad \beta_2 = t^{-1}\beta_1, \quad \varphi_{\alpha_2\beta_2} = t^2\varphi_{\alpha_1\beta_1}.$$

Even though t is not defined at the points of D, the holomorphic data is. In particular, $\varphi_{\alpha_1\beta_1}$ and $\varphi_{\alpha_2\beta_2}$ have zeroes at every point of D. Moreover, the zeroes are of the same order—this is equivalent to the measures of the corresponding limiting configurations being equal. We conclude that t is bounded around D. Since it is also (A_1, A_2)–holomorphic it extends to a holomorphic isomorphism between $(L, \bar\partial_{A_1})$ and $(L, \bar\partial_{A_2})$ and so $(A_1, \alpha_1, \beta_1, 0)$ and $(A_2, \alpha_2, \beta_2, 0)$ give rise to the same point in $\mathcal{X}$ and the map $\mathcal{X} \to \mathcal{F}$ is injective.

Step 3. *$\mathcal{X} \to \mathcal{F}$ is surjective.*

Let $(A', \Psi' = (\alpha', \beta'), D) \in \mathcal{F}$. We need to find $h \in \mathscr{G}_0^c(\Sigma\backslash D)$ such that $(A, \alpha, \beta) := (h(A'), h\alpha', h^{-1}\beta')$ extends smoothly to the whole of Σ. Furthermore, we should have $D = D_1 \cup D_2$ where D_1 and D_2 are the zero divisors of α and β respectively.

Let $\varphi_{\alpha'\beta'} \in \Gamma(\Sigma\backslash D, L^2)$ be as before. Then $\bar\partial_{A'}\varphi_{\alpha'\beta'} = 0$ and $|\varphi_{\alpha'\beta'}| = 1$ on $\Sigma\backslash D$. By Lemma 4.4.19, applied to the connection $A' \otimes A'$ and section $\varphi_{\alpha'\beta'}$, there exists $k \in \mathscr{G}_0^c(\Sigma\backslash D, L)$ such that $C := k(A' \otimes A')$ extends to a connection on a line bundle $T \to \Sigma$ and $k\varphi_{\alpha'\beta'}$ extends to a meromorphic section of $(T, \bar\partial_C)$. We claim that $T = L^2$ as unitary bundles, that $k = h^2$ for some $h \in \mathscr{G}_0^c(\Sigma\backslash D)$, and that $C = A \otimes A$ for $A = h(A')$. This follows from the assumption on the measure $i * F_{A'}$ induced by the limiting configuration (A', α', β', D); by Lemma 4.4.19 for every point $x \in D$ the meromorphic section $h^2\varphi_{\alpha'\beta'}$ vanishes to the order q_x defined by

$$\frac{i}{2\pi}F_{A'} = \frac{1}{2}\sum_{x\in D} q_x\delta_x.$$

(x is a pole of order $|q_x|$ if $q_x < 0$.) The claim is then a consequence of the assumption $\sum_{x\in D} q_x = 2d = \deg(L^2)$ and the fact that k has degree zero around the points of D. Thus, $A = h(A')$ extends. We need to show that $\alpha = h\alpha'$ and $\beta = h^{-1}\beta'$ extend. Observe that

$$|\alpha||\beta| = |\alpha'||\beta'| = \frac{1}{2}|\Psi'|^2,$$

where we have used $|\alpha'| = |\beta'|$ and $\Psi' = (\alpha', \beta')$. As a consequence, for every $x \in D$

$$|\alpha(y)||\beta(y)| = O\left(\mathrm{dist}(x,y)^{|q_x|}\right).$$

On the other hand, we have

$$\frac{|\alpha|}{|\beta|} = |h^2|\frac{|\alpha'|}{|\beta'|} = \left|h^2\varphi_{\alpha'\beta'}\right|,$$

so around $x \in D$

$$\frac{|\alpha(y)|}{|\beta(y)|} = O\left(\operatorname{dist}(x,y)^{q_x}\right).$$

We conclude that

$$|\alpha(y)| = O\left(\operatorname{dist}(x,y)^{\frac{|q_x|+q_x}{2}}\right) \qquad \text{and} \qquad |\beta(y)| = O\left(\operatorname{dist}(x,y)^{\frac{|q_x|-q_x}{2}}\right),$$

which shows that both α and β are bounded over $\Sigma\backslash D$. Since they are also holomorphic, they extend to globally defined sections. Hence, $(A,\alpha,\beta,0)$ represents a point in $\mathcal{X}$ corresponding to (A',α',β',D) under $\mathcal{X} \to \mathcal{F}$.

Step 4. *$\mathcal{X} \to \mathcal{F}$ is continuous.*

Suppose that $[A_i,\alpha_i,\beta_i] \to [A,\alpha,\beta]$ in $\mathcal{X}$. Let $(A_i',\alpha_i',\beta_i',D_i)$ and (A',α',β',D) be the corresponding points in $\mathcal{F}$. We will prove that $(A_i',\alpha_i',\beta_i',D_i)$ converges to (A',α',β',D) as limiting configurations. We easily check that the points of D_i concentrate around D and modulo simple gauge transformations

$$A_i' \to A', \qquad \alpha_i' \to \alpha', \qquad \beta_i' \to \beta'$$

in C^∞_{loc} on $\Sigma\backslash D$. By Lemma 4.4.18, $i * F_{A_i'} \to i * F_{A'}$ as measures provided that the sequence of total variations $\|i * F_{A_i'}\|$ is bounded. $\|i * F_{A_i'}\|$ is up to a constant equal to the degree of $D_1^i + D_2^i$ where D_1^i and D_2^i are the zero divisors of α_i and β_i. By Lemma 4.4.17, this degree is bounded above. We conclude that $[A_i',\alpha_i',\beta_i',D_i] \to [A',\alpha',\beta',D]$ in $\mathcal{F}$. □

4.4.4 *Proof of Theorem 4.4.1*

We construct a bijective map $\overline{\mathfrak{M}}^{\text{hol}} \to \overline{\mathfrak{M}}$ from the homeomorphisms $\mathfrak{M}^{\text{hol}} \to \mathfrak{M}$ from Theorem 4.3.6 and $\overline{\mathfrak{M}}^{\text{hol}}\backslash\mathfrak{M}^{\text{hol}} \to \mathcal{F}$ from Proposition 4.4.15. This map is continuous when restricted to $\mathfrak{M}$ and its complement. It remains to show that it is continuous; indeed, $\overline{\mathfrak{M}}^{\text{hol}}$ is compact by Proposition 4.4.4 and $\overline{\mathfrak{M}}$ is Hausdorff by Lemma 4.4.14, so a continous bijection $\overline{\mathfrak{M}}^{\text{hol}} \to \overline{\mathfrak{M}}$ is a homeomorphism.

Let $(A_i,\alpha_i,\beta_i,t_i)$ be a sequence representing points in $\mathfrak{M}^{\text{hol}}$ and (A_i',Ψ_i',t_i') the corresponding sequence of solutions of (4.4.1). Suppose that $t_i \to 0$ and (A_i,α_i,β_i) converges in C^∞ to (A,α,β) with $\alpha \neq 0$ and $\beta \neq 0$. This limit represents an element of $\overline{\mathfrak{M}}^{\text{hol}}\backslash\mathfrak{M}^{\text{hol}}$ and thus corresponds to a limiting configuration (A',Ψ',D). We need to show that after applying gauge transformations the sequence of Seiberg–Witten solutions (A_i',Ψ_i',t_i') converges in the sense of Definition 4.4.13 to a limiting configuration which is simple gauge-equivalent to (A',Ψ',D). By Theorem 4.4.6, the sequence converges and by Proposition 4.4.15, the limiting configuration is simple gauge-equivalent to (A',Ψ',D). This shows that the map $\overline{\mathfrak{M}}^{\text{hol}} \to \overline{\mathfrak{M}}$ is continuous. □

4.5 HARMONIC $\mathbf{Z}_2$ SPINORS AND COMPLEX GEOMETRY

The main result of this section, Theorem 4.5.2 below, will ensure the compactness of $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ for $\mathbf{p}$ generic among the parameters pulled back from Σ.

By the Haydys correspondence for the Seiberg–Witten equation with two spinors, cf. Section 3.4, the following definition is equivalent to the previous definition of a harmonic $\mathbf{Z}_2$ with values in $\mathrm{Re}(\mathbf{H} \otimes_{\mathbf{C}} \mathbf{C}^2)$, cf. Definition 3.1.1.

Definition 4.5.1. Let $Z \subsetneq M$ be a closed, proper subset and let

$$(A, \Psi) \in \mathscr{A}(M \backslash Z, \det W) \times \Gamma(M \backslash Z, W).$$

A triple (A, Ψ, Z) is called a *harmonic* $\mathbf{Z}_2$ *spinor with singular set* Z with respect to $\mathbf{p} = (g, B)$ if it satisfies

1. $\not{D}_{A,\mathbf{p}} \Psi = 0$ and $\mu(\Psi) = 0$,
2. $\int_{M \backslash Z} |\Psi|^2 = 1$ and $\int_{M \backslash Z} |\nabla_{A,\mathbf{p}} \Psi|^2 < \infty$,
3. $|\Psi|$ extends to a Hölder continuous function on M such that $Z = |\Psi|^{-1}(0)$.

Theorem 4.5.2. *Let $M = S^1 \times \Sigma$ be equipped with geometric data pulled back from Σ, as described at the beginning of Section 4.3.*

1. *There exists a residual subset $\mathscr{P}_\Sigma^\circ \subset \mathscr{P}_\Sigma$ with the property that for every $\mathbf{p} \in \mathscr{P}_\Sigma^\circ$ there exist no harmonic $\mathbf{Z}_2$ spinors on M with respect to $\mathbf{p}$.*
2. *For every $\mathbf{p}_0, \mathbf{p}_1 \in \mathscr{P}_\Sigma^\circ$, there exists a residual subset in the space of paths in $\mathscr{P}_\Sigma$ connecting $\mathbf{p}_0$ and $\mathbf{p}_1$ such that for every path $(\mathbf{p}_t)_{t \in [0,1]}$ from this subset there exist no harmonic $\mathbf{Z}_2$ spinors on M with respect to any $\mathbf{p}_t$.*

Fix a spin^c structure $\mathfrak{w}$ on $M = S^1 \times \Sigma$ induced from a spin structure $K^{1/2}$ on Σ and a choice of a Hermitian line bundle $L \to \Sigma$ of degree d. In the previous sections we constructed an isomorphism $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta) \cong \mathfrak{M}_d^{\mathrm{hol}}(\mathbf{p}, \eta)$. (This notation reflects the fact that $\mathfrak{M}^{\mathrm{hol}}$ depends on d and the choice of parameters of the Seiberg–Witten equation with two spinors.) Recall that $\mathfrak{M}_d^{\mathrm{hol}}(\mathbf{p}, \eta)$ contains a compact subspace $\mathfrak{N}_d(\mathbf{p}, \eta)$ introduced in Definition 4.4.3. As a consequence of Corollary 4.4.5, Theorem 4.4.6, and Theorem 4.5.2, we obtain

Corollary 4.5.3. *For a generic choice of $(\mathbf{p}, \eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$, we have*

$$\overline{\mathfrak{M}}_d^{\mathrm{hol}}(\mathbf{p}, \eta) = \mathfrak{M}_d^{\mathrm{hol}}(\mathbf{p}, \eta) = \mathfrak{N}_d(\mathbf{p}, \eta).$$

Here is an outline of the proof of Theorem 4.5.2: first, we describe harmonic $\mathbf{Z}_2$ spinors in terms of complex-geometric data on Σ. Next, we show that this data is described by a Fredholm problem of negative index, and apply the Sard–Smale theorem to exclude the existence of such data for a generic choice of $\mathbf{p} \in \mathscr{P}_\Sigma$. For the remaining part of this section, we continue to assume that $M = S^1 \times \Sigma$ and $(\mathbf{p}, \eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$.

4.5.1 S^1–*invariance of harmonic* $\mathbf{Z}_2$ *spinors*

Let (A, Ψ, Z) be a harmonic $\mathbf{Z}_2$ spinor as in Definition 4.5.1 (equivalently, a solution of the limiting Seiberg–Witten equation with two spinors). Suppose that $Z = S^1 \times D$ for $D \subset \Sigma$, and that (A, Ψ) is pulled back from Σ. Then, as in Section 4.3, we have $\Psi = (\alpha, \bar{\beta})$ where

$$\alpha \in \Gamma(\Sigma \backslash D, E^* \otimes L \otimes K^{1/2}),$$

$$\beta \in \Gamma(\Sigma \backslash D, E \otimes L^* \otimes K^{1/2}).$$

The Fueter equations $\not{D}_{AB}\Psi = 0$ and $\mu(\Psi) = 0$ are equivalent to

$$\begin{cases} \bar{\partial}_{AB}\alpha = 0, \quad \bar{\partial}_{AB}\beta = 0, \\ \alpha\beta = 0, \\ |\alpha| = |\beta|. \end{cases} \tag{4.5.1}$$

Proposition 4.5.4. *Let (A, Ψ, Z) be a harmonic Z_2 spinor on $M = S^1 \times \Sigma$ with respect to $\mathbf{p} \in \mathscr{P}_\Sigma$. Then $Z = S^1 \times D$ for a finite subset $D \subset \Sigma$. Moreover, there is a gauge transformation $u \in \mathscr{G}(M \backslash Z)$ such that u has degree zero around each component of Z and $u(A, \Psi)$ is pulled back from a configuration on $\Sigma \backslash D$.*

Proof. The proof is similar to that of Theorem 4.1.12. We use the notation from Section 4.1, with one minor convention change: Ψ denotes here what in Section 4.1 we called $\overline{\Psi}$. As in the proof of Theorem 4.1.12, we assume for simplicity that the background connection B is trivial; the general proof is the same.

Step 1. *A Weitzenböck formula.*

Let t be the coordinate on the S^1 factor of $S^1 \times \Sigma$. Unlike in the proof of Theorem 4.1.12, we cannot put A in a temporal gauge, even after pulling back to $\mathbf{R} \times \Sigma$, because *a priori* the singular set Z could intersect the t–axis in a complicated way. However, by Proposition 4.1.7, we still have

$$0 = \not{D}_A\Psi = -i\sigma\nabla_t\Psi + \sqrt{2}\bar{\partial}_A\Psi, \tag{4.5.2}$$

where $\nabla_t = \nabla_A(\partial/\partial t)$ and $\bar{\partial}_A$ is the Dolbeault operator induced by A on the $\{t\} \times \Sigma$ slice. Let ∇_Σ be the part of the covariant derivative ∇_A in the Σ–direction. Since A is flat, ∇_t and ∇_Σ commute. Applying σ and $\nabla_t^* = -\nabla_t$ to (4.5.2) and using the above commutation relation, as well as the identity $\sigma\bar{\partial}_A\sigma = \bar{\partial}_A^*$ from Remark 4.1.9, we obtain

$$0 = \nabla_t^*\nabla_t\Psi + 2\sigma\bar{\partial}_A\sigma\bar{\partial}_A\Psi = \nabla_t^*\nabla_t\Psi + 2\bar{\partial}_A^*\bar{\partial}_A\Psi. \tag{4.5.3}$$

Step 2. *Integration by parts; the circle-invariance of Z.*

We want to integrate (4.5.3) by parts to conclude $\nabla_t\Psi = 0$ and $\bar{\partial}_A\Psi = 0$. The equality holds only on $M \backslash Z$, so we need to use a cut-off function. Let $f\colon \mathbf{R} \to [0,1]$ be smooth and such that

$$\begin{cases} f = 0 & \text{on } (-\infty, 0], \\ f = 1 & \text{on } [1, \infty) \end{cases}$$

For every $\varepsilon > 0$ we define the cut-off function $\chi_\varepsilon \colon M \to [0,1]$ by

$$\chi_\varepsilon(x) = f\left(\frac{|\Psi(x)| - \varepsilon}{\varepsilon}\right).$$

Let Z_ε be the subset of points in M satisfying $|\Psi(x)| < \varepsilon$. We have

$$\begin{cases} \chi_\varepsilon = 0 & \text{on } Z_\varepsilon, \\ \chi_\varepsilon = 1 & \text{on } M \backslash Z_{2\varepsilon} \end{cases}$$

and χ_ε is smooth on M. Take the inner product of (4.5.3) with $\chi_\varepsilon^2 \Psi$ and integrate by parts:

$$\begin{aligned} 0 &= \frac{1}{2}\int_M |\nabla_t(\chi_\varepsilon \Psi)|^2 + \int_M |\bar{\partial}_A(\chi_\varepsilon \Psi)|^2 - \int_M \left(|\partial_t \chi_\varepsilon|^2 + |\bar{\partial}\chi_\varepsilon|^2\right)|\Psi|^2 \\ &\geqslant \frac{1}{2}\int_M |\nabla_t(\chi_\varepsilon \Psi)|^2 + \int_M |\bar{\partial}_A(\chi_\varepsilon \Psi)|^2 - 2\int_M |\mathrm{d}\chi_\varepsilon|^2 |\Psi|^2. \end{aligned} \tag{4.5.4}$$

We need to show that the last term becomes arbitrarily small as ε tends to zero. By definition, $|\Psi| \leqslant 2\varepsilon$ on $Z_{2\varepsilon}$. Let $P_\varepsilon = Z_{2\varepsilon} \backslash Z_\varepsilon$. By Kato's inequality

$$\begin{aligned} \int_M |\Psi|^2 |d\chi_\varepsilon|^2 &\leqslant \int_{P_\varepsilon} |\Psi|^2 \frac{\left| f'\left(\frac{|\Psi(x)| - \varepsilon}{\varepsilon}\right)\right|^2}{\varepsilon^2} |\nabla_A \Psi|^2 \\ &\leqslant 4\|f\|_{C^1}^2 \int_{P_\varepsilon} |\nabla_A \Psi|^2 \\ &\leqslant C\mathrm{vol}(P_\varepsilon)\|\nabla_A \Psi\|^2_{L^2(M\backslash Z)}. \end{aligned}$$

The right-hand side converges to zero as $\varepsilon \to 0$ since $|\nabla_A \Psi|^2$ is integrable, $Z = \cap_{\varepsilon > 0} Z_\varepsilon$, and $\mathrm{vol}(Z) = 0$, as proved by Taubes [Tau14, Theorem 1.3]. Taking $\varepsilon \to 0$ in (4.5.4), we conclude that on $M \backslash Z$

$$\nabla_t \Psi = 0 \qquad \text{and} \qquad \bar{\partial}_A \Psi = 0.$$

In particular,

$$\partial_t |\Psi|^2 = 2\langle \nabla_t \Psi, \Psi \rangle = 0$$

so $|\Psi|$ is invariant under the S^1–action on $M \backslash Z$. By assumption, Ψ is continuous on the whole of M and $|\Psi|^{-1}(0) = Z$, so Z is necessarily of the form $S^1 \times D$ for a proper subset $D \subset \Sigma$.

Step 3. *(A, Ψ) is pulled back from $\Sigma \backslash D$.*

We put A in a temporal gauge over $S^1 \times (\Sigma \backslash D)$ as in the proof of Theorem 4.1.12. The gauge transformation (4.1.8) used to do that is the exponential of a smooth function $\Sigma \backslash D \to i\mathbf{R}$ when restricted to each slice $\{t\} \times (\Sigma \backslash D)$; thus, it has degree zero around the components of Z. The same argument as in the proof of Theorem 4.1.12 shows that $L_{|M\backslash Z}$ is pulled back from a bundle on $\Sigma \backslash D$ and (A, Ψ) is pulled back from a configuration on $\Sigma \backslash D$ satisfying (4.5.1).

Step 4. *D is a finite set of points.*

It is enough to show that D is locally finite. Suppose that Σ is a unit disc and that L and E are trivial. The complement $\Sigma\backslash D$ is a non-compact Riemann surface and $(L, \bar{\partial}_A)$ defines a holomorphic line bundle over $\Sigma\backslash D$ which is necessarily trivial [For91, Theorem 30.3]. Thus, there is $h \in \mathscr{G}^c(B\backslash D)$ such that $h(A)$ agrees with the product connection on the trivial bundle, and $h\alpha$ and $h^{-1}\beta$ correspond to holomorphic maps $\Sigma\backslash D \to \mathbf{C}^2$. Let $\gamma = (h\alpha) \otimes (h^{-1}\beta)$; it is a holomorphic map $\Sigma\backslash D \to \mathbf{C}^2 \otimes \mathbf{C}^2 = \mathbf{C}^4$ satisfying

$$|\gamma| = |\alpha||\beta| = \frac{1}{2}|\Psi|^2,$$

so D is the zero set of $|\gamma|$. Thus, γ is continuous on Σ and holomorphic on $\Sigma\backslash D$. By a theorem of Radó [Rud87, Theorem 12.14], γ is holomorphic on Σ and so $D = \gamma^{-1}(0)$ is locally finite. □

4.5.2 *A holomorphic description of harmonic $\mathbf{Z}_2$ spinors*

Proposition 4.5.5. *If (A, Ψ, Z) is a harmonic $\mathbf{Z}_2$ spinor as in Proposition 4.5.4, with $\Psi = (\alpha, \bar{\beta})$ and $Z = S^1 \times D$, then there exist $h \in \mathscr{G}_0^c(\Sigma\backslash D)$ and divisors D_1, D_2 on Σ such that*

1. *The divisor $D_1 + D_2$ is effective, and $D = D_1 \cup D_2$ as sets.*
2. *$\tilde{A} := h(A)$ extends to a unitary connection on a line bundle over Σ, not necessarily isomorphic to L, defining a holomorphic line bundle $\mathscr{L} \to \Sigma$,*
3. *Sections $\tilde{\alpha} = h\alpha$ and $\tilde{\beta} = h^{-1}\beta$ extend to holomorphic sections over all of Σ, which fit into the short exact sequence*

$$0 \longrightarrow \mathscr{L}^{-1} \otimes K^{-1/2} \otimes \mathcal{O}(D_1) \xrightarrow{\tilde{\alpha}} \mathscr{E} \xrightarrow{\tilde{\beta}} \mathscr{L}^{-1} \otimes K^{1/2} \otimes \mathcal{O}(-D_2) \longrightarrow 0. \tag{4.5.5}$$

Conversely, every set of holomorphic data $(\tilde{A}, \tilde{\alpha}, \tilde{\beta}, D_1, D_2)$ satisfying conditions (1), (2), (3) can be obtained from a harmonic $\mathbf{Z}_2$ spinor (A, Ψ, Z) in this way.

Proof. This is similar to Step 3 in the proof of Proposition 4.4.15. Using Lemma 4.4.19, we find $h \in \mathscr{G}_0^c(\Sigma\backslash D, L)$ such that $\tilde{A} = h(A)$ extends yielding a holomorphic line bundle $\mathscr{L}$, say, and $h^2\varphi_{\alpha\beta}$ extends to a meromorphic section of $\mathscr{L}^2$. Let $\tilde{\alpha} = h\alpha$ and $\tilde{\beta} = h^{-1}\beta$. Then

$$\frac{|\tilde{\alpha}|}{|\tilde{\beta}|} = |h^2\varphi_{\alpha\beta}| \qquad \text{and} \qquad |\tilde{\alpha}||\tilde{\beta}| = |\alpha||\beta| = \frac{1}{2}|\Psi|^2.$$

Since $h^2\varphi_{\alpha\beta}$ is meromorphic and $|\Psi|$ extends to a continuous function on Σ, it follows that $\tilde{\alpha}$ and $\tilde{\beta}$ extend to meromorphic sections. Let D_1 and D_2 be the associated divisors of zeroes and poles. We have $D = D_1 \cup D_2$ as sets and the condition $D = |\Psi|^{-1}(0)$ implies that $D_1 + D_2 \geqslant 0$. The existence of the short exact sequence involving $\tilde{\alpha}$ and $\tilde{\beta}$ was established in (4.4.2). □

The next lemma provides a restriction on the possible holomorphic bundles $\mathscr{E}$ fitting into the short exact sequence (4.5.5).

Lemma 4.5.6. *Under the assumptions of Proposition 4.5.5 there exists a holomorphic line bundle $\mathscr{T}$ satisfying $h^0(\mathscr{T}^2) > 0$ and $h^0(\mathscr{E} \otimes K^{1/2} \otimes \mathscr{T}^{-1}) > 0$.*

Proof. Recall that by Lemma 4.4.16 we have $\mathscr{L}^2 = \mathcal{O}(D_1 - D_2)$. Set $\mathscr{T} = \mathscr{L}^{-1} \otimes \mathcal{O}(D_1)$. Then

$$\mathscr{T}^2 = \mathscr{L}^{-2} \otimes \mathcal{O}(2D_1) = \mathcal{O}(D_2 - D_1 + 2D_1) = \mathcal{O}(D_1 + D_2).$$

We have $h^0(\mathscr{T}^2) > 0$ because the divisor $D_1 + D_2$ is effective. On the other hand, multiplying exact sequence (4.5.5) by $\mathscr{T}^{-1} \otimes K^{1/2}$, we obtain an injective map $\mathcal{O} \to \mathscr{E} \otimes \mathscr{T}^{-1} \otimes K^{1/2}$, which is the same as a nowhere vanishing section of $\mathscr{E} \otimes K^{1/2} \otimes \mathscr{T}^{-1}$. □

4.5.3 *Proof of Theorem 4.5.2*

Lemma 4.5.7. *Fix $k \geqslant 0$ and a conformal structure on Σ. For every $B \in \mathscr{A}(\Sigma, E)$, let $\mathscr{E}_B$ denote the holomorphic bundle $(E, \bar\partial_B)$.*

Let $\mathcal{Z} \subset \mathscr{A}(\Sigma, E)$ be the subset consisting of those connections B for which there exists a degree k holomorphic line bundle $\mathscr{T} \to \Sigma$ satisfying

$$h^0(\mathscr{T}^2) > 0 \quad \text{and} \quad h^0(\mathscr{E}_B \otimes K^{1/2} \otimes \mathscr{T}^{-1}) > 0.$$

The complement $\mathscr{A}(\Sigma, E) \backslash \mathcal{Z}$ is residual in $\mathscr{A}(\Sigma, E)$. Furthermore, for all $B_0, B_1 \in \mathscr{A}(\Sigma, E) \backslash \mathcal{Z}$, a generic path in $\mathscr{A}(\Sigma, E)$ connecting B_0 and B_1 is disjoint from $\mathcal{Z}$.

Proof. We use a Sard–Smale argument similar to the one used in the proof of Proposition 3.2.6. As in that proof, in the argument we replace the spaces of smooth connections and sections by their suitable Sobolev completions.

Let $T \to \Sigma$ be a Hermitian line bundle of degree k. Denote

$$F = E^* \otimes K^{1/2} \otimes T^{-1}$$

and consider the map

$$\mathscr{A}(\Sigma, E) \times \mathscr{A}(\Sigma, T) \times \Gamma(F) \times \Gamma(T^2) \to \Omega^{0,1}(F) \times \Omega^{0,1}(T^2),$$
$$(B, A, \psi, \alpha) \mapsto (\bar\partial_{AB}\psi, \bar\partial_A \alpha).$$

This map is $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$–equivariant. Let $\mathcal{X}$ be the open subset of $\mathscr{A}(\Sigma, T) \times \Gamma(F) \times \Gamma(T^2)/\mathscr{G}^{\mathbf{C}}(\Sigma, L)$ given by

$$\mathcal{X} = \{[B, A, \psi, \alpha] \mid \psi \neq 0,\ \alpha \neq 0\}.$$

Let $\mathcal{V} \to \mathscr{A}(\Sigma, E) \times \mathcal{X}$ be the Banach vector bundle obtained from taking the $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$–quotient of the trivial bundle with fiber $\Omega^{0,1}(F) \times \Omega^{0,1}(T)$. Then the map introduced above descends to a smooth section $s\colon \mathscr{A}(\Sigma, E) \times \mathcal{X} \to \mathcal{V}$. For every $B \in \mathscr{A}(\Sigma, E)$ the restriction $s_B = s(B, \cdot)$ is a Fredholm section whose index is the Euler characteristic of the elliptic complex

$$\Omega^0(\mathbf{C}) \longrightarrow \Omega^{0,1}(\mathbf{C}) \oplus \Gamma(F) \oplus \Gamma(T^2) \longrightarrow \Omega^{0,1}(F) \oplus \Omega^{0,1}(T^2). \qquad (4.5.6)$$

The first arrow in the complex is the linearized action of $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$, whereas the second is the linearization of the map $(A, \psi, \alpha) \mapsto (\bar\partial_{AB}\psi, \bar\partial_A\alpha)$. This elliptic complex agrees up to terms of order zero with the direct sum of the complexes

$$\Omega^0(\mathbf{C}) \xrightarrow{\bar\partial} \Omega^{0,1}(\mathbf{C}) \longrightarrow 0 \qquad \text{and}$$

$$0 \longrightarrow \Gamma(F) \oplus \Gamma(T^2) \xrightarrow{\bar\partial_{AB} \oplus \bar\partial_A} \Omega^{0,1}(F) \oplus \Omega^{0,1}(T^2).$$

By the Riemann–Roch theorem, the Euler characteristic of this complex is

$$\chi(\mathcal{O}) - \chi(F) - \chi(T^2) = (1-g) - (\deg(F) + 2 - 2g) - (2\deg(T) + 1 - g) = 0$$

because $\deg(F) = 2g - 2 - 2\deg(T)$. Thus, s_B is a Fredholm section of index zero.

The proof will be completed if we can show that s is transverse to the zero section at all points $[B, A, \psi, \alpha] \in s^{-1}(0) \subset \mathcal{X}$. Indeed, if this is the case, then by the Sard–Smale theorem, the same is true for s_B for B from a residual subset of $\mathscr{A}(\Sigma, E)$. For every such B the set

$$\{[A, \psi, \alpha] \mid \bar\partial_{AB}\psi = 0,\ \bar\partial_A \alpha = 0,\ \psi \neq 0\ \alpha \neq 0\}$$

is a zero-dimensional submanifold of $\mathcal{X}$. This submanifold must be empty as otherwise it would contain a subset homeomorphic to $\mathbf{C}^*$ given by $[A, t\psi, \alpha]$ for $t \in \mathbf{C}^*$. This proves that for a generic B there is no holomorphic line bundle $\mathscr{T} = (T, \bar\partial_A)$ together with non-zero $\alpha \in H^0(\mathscr{T}^2)$ and $\psi \in H^0(\mathscr{E}_B \otimes K^{1/2} \otimes \mathscr{T}^{-1})$. The statement for paths is proved in the same way.

It remains to show that s is transverse to the zero section. At a point $[B, A, \psi, \alpha] \in s^{-1}(0)$ the first map in (4.5.6) is injective. Thus, it is enough to prove the surjectivity of the operator combining the second map of (4.5.6) and the linearization of $\bar\partial_{AB}$ with respect to B:

$$\Omega^{0,1}(\mathrm{End}(F)) \oplus \Omega^{0,1}(\mathbf{C}) \oplus \Gamma(F) \oplus \Gamma(T^2) \longrightarrow \Omega^{0,1}(F) \oplus \Omega^{0,1}(T^2)$$

$$(b, a, u, v) \mapsto ((b+a)\psi + \bar\partial_{AB} u, a\alpha + \bar\partial_A v).$$

If this map were not surjective, there would exist a non-zero $(p, q) \in \Omega^{0,1}(F) \oplus \Omega^{0,1}(T^2)$ L^2–orthogonal to the image; which in turn would imply $\bar\partial^*_{AB} p = 0$, $\bar\partial^*_A q = 0$, and

$$\langle b\psi, p \rangle_{L^2} = 0 \quad \text{and} \quad \langle a\alpha, q \rangle_{L^2} = 0$$

for all $b \in \Omega^{0,1}(\mathrm{End}(F))$ and $a \in \Omega^{0,1}(\mathbf{C})$. Note that ψ and α are both non-zero and holomorphic; p and q are anti-holomorphic and at least one of them is non-zero. Thus, using a bump function, we can construct b and a such that

$$\langle b\psi, p \rangle_{L^2} > 0 \quad \text{and} \quad \langle a\alpha, q \rangle_{L^2} > 0.$$

The contradiction shows that s is transverse to the zero section. □

Theorem 4.5.2 follows immediately from the previous results. Let $\mathbf{p} = (g, B) \in \mathscr{P}_\Sigma$. Denote by $\mathscr{E}_B$ the holomorphic bundle $(E, \bar\partial_B)$. Proposition 4.5.4 and Proposition 4.5.5 show that a harmonic $\mathbf{Z}_2$ spinor with respect to $\mathbf{p}$ corresponds to a holomorphic triple $(\mathscr{L}, \alpha, \beta)$ fitting into the short exact sequence (4.5.5). On the other hand, by Lemma 4.5.6 and Lemma 4.5.7, given any g, for a generic choice of B the holomorphic bundle $\mathscr{E}_B$ does not fit into any such sequence. The same is true when (g, B) vary in a generic one-parameter family by the second part of Lemma 4.5.7. □

4.5.4 *Harmonic $\mathbf{Z}_2$ spinors and limiting configurations*

Every limiting configurations, as in Definition 4.4.10, is an example of a harmonic $\mathbf{Z}_2$ spinor on $M = S^1 \times \Sigma$. The results established in this section allow us to construct an example of a harmonic $\mathbf{Z}_2$ spinor which is not a limiting configuration. This shows that the moduli space $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is compactified using only some, but not all, harmonic $\mathbf{Z}_2$ spinors.

Example 4.5.8. Suppose that the genus of Σ is positive so that the canonical divisor K is effective. Let C_1 and C_2 be two divisors satisfying $C_1 + C_2 = \frac{1}{2}K \geqslant 0$. Set

$$\mathscr{L} := \mathcal{O}(C_1 - C_2), \qquad D_1 := 2C_1, \qquad D_2 := 2C_2,$$

Then

$$\mathscr{L}^{-1} \otimes K^{-1/2} \otimes \mathcal{O}(D_1) = \mathcal{O}(C_1 + C_2 - \frac{1}{2}K) = \mathcal{O},$$

$$\mathscr{L}^{-1} \otimes K^{1/2} \otimes \mathcal{O}(-D_2) = \mathcal{O}(-C_1 - C_2 + \frac{1}{2}K) = \mathcal{O}.$$

Set $\mathscr{E} := \mathcal{O} \oplus \mathcal{O}$. According to Proposition 4.5.5, if we can find holomorphic maps $\tilde{\alpha}$ and $\tilde{\beta}$ making the sequence (4.5.5) exact, then the data $(\mathscr{L}, D_1, D_2, \tilde{\alpha}, \tilde{\beta})$ gives rise to a harmonic $\mathbf{Z}_2$ spinor with singular set $D = D_1 \cup D_2$.

In the present setting, the sequence (4.5.5) is

$$0 \longrightarrow \mathcal{O} \xrightarrow{\tilde{\alpha}} \mathcal{O} \oplus \mathcal{O} \xrightarrow{\tilde{\beta}} \mathcal{O} \longrightarrow 0,$$

so there is an obvious choice of $\tilde{\alpha}$ and $\tilde{\beta}$ making the sequence exact. However,the harmonic $\mathbf{Z}_2$ spinor corresponding to $(\mathscr{L}, D_1, D_2, \tilde{\alpha}, \tilde{\beta})$ is not a limiting configuration unless both divisors C_1 and C_2 are effective. For if C_1 or C_1 is not effective, the last condition in Definition 4.4.9 is violated.

4.6 Moduli spaces of framed vortices

We continue to assume that $M = S^1 \times \Sigma$ and that the spin^c structure $\mathfrak{w}$ is induced from a spin structure $K^{1/2} \to \Sigma$ and a Hermitian line bundle $L \to \Sigma$ of degree d.

By Theorem 4.5.2 and Corollary 4.5.3, for a generic choice of S^1–invariant parameters $(\mathbf{p}, \eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$, the moduli space $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ is homeomorphic to the compact space $\mathfrak{N}_d(\mathbf{p}, \eta)$ introduced in Definition 4.4.3. In this section we prove that in this situation $\mathfrak{N}_d(\mathbf{p}, \eta)$ is a Kähler manifold and that the signed count of monopoles with two spinors on $M = S^1 \times \Sigma$ is the signed Euler characteristic of $\mathfrak{N}_d(\mathbf{p}, \eta)$. As shown by Theorem 3.5.7, the signed count of monopoles with two spinors depends, in general, on the choice of $(\mathbf{p}, \eta) \in \mathscr{P} \times \mathscr{Z}$. However, it turns out that it is the same for all generic choices of $(\mathbf{p}, \eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$. As we will see, the reason is that harmonic $\mathbf{Z}_2$ spinors appear for $\mathbf{p}$ from a subset in $\mathscr{P}_\Sigma$ of real codimension two.

Theorem 4.6.1. *Let Σ be a closed spin surface of genus $g(\Sigma) \geqslant 1$. Equip $M = S^1 \times \Sigma$ with a spin^c structure $\mathfrak{w}$ induced from the spin structure $K^{1/2}$ on Σ and a Hermitian line bundle $L \to \Sigma$. Denote by $d = \langle c_1(L), [\Sigma] \rangle$ the degree of L.*

For a generic choice of $(\mathbf{p}, \eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$ *there exist no harmonic* $\mathbf{Z}_2$ *spinors with respect to* $\mathbf{p}$ *and the moduli space* $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ *is a compact Kähler manifold of dimension*

$$\dim_{\mathbf{C}} \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta) = g(\Sigma) - 1 \pm 2d,$$

where $\pm = \operatorname{sign}(i/2\pi \int_\Sigma *\eta - d)$*. In this case, the signed count of Seiberg–Witten multi-monopoles is, up to a sign, the Euler characteristic of the moduli space:*

$$n_{\mathfrak{w}}(\mathbf{p}, \eta) = (-1)^{g(\Sigma)-1} \chi(\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)).$$

Moreover, $n_{\mathfrak{w}}(\mathbf{p}, \eta)$ *does not depend on the choice of a generic* $(\mathbf{p}, \eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$*.*

In order to prove Theorem 4.6.1, we study the moduli space $\mathfrak{N}_d(\mathbf{p}, \eta)$.

4.6.1 *Framed vortices*

We assume throughout this section that $d - \tau < 0$ where $d = \deg L$ and $\tau = \tau(\eta) = \int_\Sigma i\eta/2\pi$. All the results of this section can be easily generalized to the case $d - \tau > 0$. The case $d - \tau = 0$ is uninteresting as the moduli space is generically empty.

The space $\mathfrak{N}_d(\mathbf{p}, \eta)$ was introduced in introduced in Definition 4.4.3. Its points can be interpreted in three different ways.

1. As isomorphism classes of pairs $(\mathscr{L}, \alpha)$, where $\mathscr{L} \to \Sigma$ is a degree d holomorphic line bundle and α is a non-zero holomorphic section

   $$\alpha \in H^0(\Sigma, \mathscr{E}_B \otimes \mathscr{L} \otimes K^{1/2}).$$

   Recall that $\mathscr{E}_B$ denotes the holomorphic bundle $(E, \bar{\partial}_B)$, where B is the given connection on E, part of the parameter $\mathbf{p} = (g, B)$.

2. As $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$–equivalence classes of pairs

   $$(A, \alpha) \in \mathscr{A}(\Sigma, L) \times \Gamma(\Sigma, E^* \otimes L \otimes K^{1/2})$$

   satisfying $\bar{\partial}_{AB}\alpha = 0$ and $\alpha \neq 0$.

3. As $\mathscr{G}(\Sigma, L)$–equivalence classes of pairs (A, α) as above satisfying

   $$\begin{cases} \bar{\partial}_{AB}\alpha = 0 \quad \text{and } \alpha \neq 0, \\ i * F_A + |\alpha|^2 - i * \eta = 0, \end{cases} \tag{4.6.1}$$

Following [BGP97], we will refer to $\mathfrak{N}_d(\mathbf{p}, \eta)$ as the *moduli space of framed vortices* (in this context, *framing* corresponds to fixing the holomorphic bundle $\mathscr{E}_B$).

4.6.2 *Deformation theory*

Here we relate the deformation theories of $\mathfrak{N}_d(\mathbf{p}, \eta)$ and $\mathfrak{M}_d^{\text{hol}}(\mathbf{p}, \eta)$.

Theorem 4.6.2. *Under the assumptions of Theorem 4.6.1, for a generic choice of* $(\mathbf{p}, \eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$ *the following hold:*

1. $\mathfrak{N}_d(\mathbf{p}, \eta)$ *is a compact Kähler manifold of complex dimension* $g(\Sigma) - 1 \pm 2d$*, where* $\pm = \operatorname{sign}(\tau(\eta) - d)$.

2. $\mathfrak{M}_d^{\mathrm{hol}}(\mathbf{p},\eta)$ *is Zariski smooth in the sense of Definition 3.2.17. Moreover, the inclusion*

$$\mathfrak{N}_d(\mathbf{p},\eta) \hookrightarrow \mathfrak{M}_d^{\mathrm{hol}}(\mathbf{p},\eta)$$

is a homeomorphism and induces an isomorphism of Zariski tangent spaces.

3. *The relative orientation on the obstruction bundle* $\mathfrak{O} \to \mathfrak{M}_d^{\mathrm{hol}}(\mathbf{p},\eta)$*, introduced in Proposition 3.2.18, is compatible with the orientation of the cotangent bundle* $T^*\mathfrak{N}_d(\mathbf{p},\eta) \to \mathfrak{N}_d(\mathbf{p},\eta)$ *induced from the complex structure.*

Before proving the theorem, let us discuss some of its consequences.

Corollary 4.6.3. *The following conditions are equivalent:*

1. *All points of* $\mathfrak{N}_d(\mathbf{p},\eta)$ *are unobstructed in the sense of deformation theory of solutions to the framed vortex equation (4.6.1).*
2. $\mathfrak{M}_d^{\mathrm{hol}}(\mathbf{p},\eta)$ *is compact and* $\mathfrak{M}_d^{\mathrm{hol}}(\mathbf{p},\eta) = \mathfrak{N}_d(\mathbf{p},\eta)$.
3. *There exist no triple* $(\mathscr{L},\alpha,\beta)$ *consisting of a degree d holomorphic line bundle* $\mathscr{L} \to \Sigma$ *and non-zero holomorphic sections* $\alpha \in H^0(\Sigma, \mathscr{E}_B \otimes \mathscr{L} \otimes K^{1/2})$ *and* $\beta \in H^0(\Sigma, \mathscr{E}_B^* \otimes \mathscr{L}^* \otimes K^{1/2})$ *satisfying* $\alpha\beta = 0 \in H^0(\Sigma, K)$.

Proof. The equivalence of (1) and (2) follows from Corollary 4.4.5 and the identification of the obstruction bundle $\mathfrak{O} = \bigcup_{A,\alpha} H^2_{A,\alpha}$ with $\overline{\mathfrak{M}}_d^{\mathrm{hol}}(\mathbf{p},\eta)\backslash\mathfrak{M}_d^{\mathrm{hol}}(\mathbf{p},\eta)$, shown in the proof of Theorem 4.6.2. The equivalence of (2) and (3) is obvious from the definition of $\overline{\mathfrak{M}}^{\mathrm{hol}}$. □

Corollary 4.6.4. *The diffeomorphism type of* $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p},\eta)$ *does not depend on a generic choice of* $(\mathbf{p},\eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$*, as long as we vary* η *so that* $\operatorname{sign}(d-\tau(\eta))$ *remains constant.*

Proof. Let $\mathbf{p}_0, \mathbf{p}_1 \in \mathscr{P}_\Sigma^\circ$, where $\mathscr{P}_\Sigma^\circ \subset \mathscr{P}_\Sigma$ is a residual set from Theorem 4.5.2. For a generic path $(\mathbf{p}_t)_{t\in[0,1]}$ in $\mathscr{P}_\Sigma$ connecting $\mathbf{p}_0$ and $\mathbf{p}_1$, there exist no harmonic $\mathbf{Z}_2$ spinors with respect to $\mathbf{p}_t$, for any t. Thus, for any path $(\eta_t)_{t\in[0,1]}$ in $\mathscr{Z}_\Sigma$ such that $\operatorname{sign}(d-\tau(\eta_t))$ remains constant, $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}_t,\eta_t) = \mathfrak{N}_d(\mathbf{p}_t,\eta_t)$ is compact consists of irreducible, unobstructed solutions to (4.6.1). (If, for some t, we have $d = \tau(\eta_t)$, then reducible solutions appear.) Thus, $\bigcup_{t\in[0,1]} \mathfrak{N}_d(\mathbf{p}_t,\eta_t) \to [0,1]$ is a smooth fiber bundle and all of its fibers are diffeomorphic. □

In the remaining part of this section, we prove Theorem 4.6.2 and show that it implies Theorem 4.6.1. Choose $(\mathbf{p},\eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$ so that there exist no harmonic $\mathbf{Z}_2$ spinors with respect to $\mathbf{p}$, and the moduli space $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p},\eta)$ consists of irreducible and unobstructed monopoles. We know that this is true for a generic choice of $(\mathbf{p},\eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$. To simplify the notation, in what follows denote

$$\begin{aligned} \mathfrak{N} &= \mathfrak{N}_d(\mathbf{p},\eta), \\ \mathfrak{M}^{\mathrm{hol}} &= \mathfrak{M}_d^{\mathrm{hol}}(\mathbf{p},\eta), \\ \mathfrak{M} &= \mathfrak{M}_{\mathfrak{w}}(\mathbf{p},\eta). \end{aligned}$$

Proof of Theorem 4.6.2. The proof proceeds in four steps.

Step 1. *Deformation theory of* $\mathfrak{N}$.

The construction of an analytic structure on $\mathfrak{N}$ follows the general scheme that by now is familiar to the reader. Consider the elliptic complex associated to a solution (A,α) of $\bar\partial_{AB}\alpha = 0$:

$$\Omega^0(\mathbf{C}) \xrightarrow{G^c_{A,\alpha}} \Omega^{0,1}(\mathbf{C}) \oplus \Gamma(E^* \otimes L \otimes K^{1/2}) \xrightarrow{T_{A,\alpha}} \Omega^{0,1}(E^* \otimes L \otimes K^{1/2}).$$

where $G^c_{A,\alpha}$ is the linearized action of $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$

$$G^c_{A,\alpha}(f) = (-\bar\partial f, f\alpha) \qquad \text{for } f \in \Omega^0(\mathbf{C}),$$

and $T_{A,\alpha}$ is the linearization of the Dolbeault operator

$$T_{A,\alpha}(a^{0,1}, \phi) = (\bar\partial_{AB}\phi + a^{0,1}\alpha) \qquad \text{for } (a^{0,1}, \phi) \in \Omega^{0,1}(\mathbf{C}) \oplus \Gamma(E^* \otimes L \otimes K^{1/2}).$$

Denote by $H^0_{A,\alpha}$, $H^1_{A,\alpha}$, and $H^2_{A,\alpha}$ the homology groups of this complex. By definition $\mathfrak{N}$ consists of solutions with $\alpha \neq 0$, so $H^0_{A,\alpha} = 0$. On the other hand, the deformation complex is isomorphic modulo lower order term to the sum of the Dolbeault complexes for $\bar\partial$ on $\Omega^0(\mathbf{C})$ and $\bar\partial_{AB}$ on $E^* \otimes L \otimes K^{1/2}$ (with a shift). By the Riemann–Roch theorem the expected complex dimension of $\mathfrak{N}$ is

$$\dim_{\mathbf{C}} H^1_{A,\alpha} - \dim_{\mathbf{C}} H^2_{A,\alpha} = \chi(\Sigma, \mathcal{O}) - \chi(\Sigma, \mathscr{E} \otimes \mathscr{L} \otimes K^{1/2}) = g(\Sigma) - 1 + 2d.$$

Step 2. *$\mathfrak{N}$ is a compact Kähler manifold.*

We already know that $\mathfrak{N}$ is compact. By Corollary 4.5.3, $\mathfrak{M}^{\text{hol}} = \mathfrak{N}$ for a generic B. One can show that $\mathfrak{N}$ is generically smooth in the same way as in Proposition 3.2.6. Alternatively, we can interpret the elements of $H^2_{A,\alpha}$ as harmonic $\mathbf{Z}_2$ spinors:

$$H^2_{A,\alpha} = \ker T^*_{A,\alpha} = \left\{ q \in \Gamma(E^* \otimes L \otimes K^{-1/2}) \mid \bar\alpha q = 0,\ \bar\partial^*_{AB} q = 0 \right\}.$$

Every non-zero element of $H^2_{A,\alpha}$ gives rise to a non-zero $\beta = \bar q \in \Gamma(E \otimes L^* \otimes K^{1/2})$ satisfying $\bar\partial_{AB}\beta = 0$ and $\alpha\beta = 0$. Thus, the triple (A, α, β) is an element of $\overline{\mathfrak{M}}^{\text{hol}} \backslash \mathfrak{M}^{\text{hol}}$ corresponding to a harmonic $\mathbf{Z}_2$ spinor as in Proposition 4.4.15. By Theorem 4.5.2, for a generic B there are no harmonic $\mathbf{Z}_2$ spinors so $H^2_{A,\alpha} = 0$ for all $[A,\alpha] \in \mathfrak{N}$. This implies that $\mathfrak{N}$ is a complex manifold of dimension $g(\Sigma) - 1 + 2d$ whose holomorphic tangent space at $[A,\alpha]$ is $H^1_{A,\alpha}$. It admits a natural Hermitian metric induced from the L^2–inner product on the space of connections and sections. This metric is Kähler because $\mathfrak{N}$ is the moduli space of solutions of the framed vortex equations (4.6.1), which is an infinite-dimensional Kähler quotient. For details, see [Per08; DT16].

Step 3. *$H^1_{A,\alpha}$ is naturally isomorphic to the Zariski tangent space to $\mathfrak{M}^{\text{hol}}$ at $[A, \alpha, 0]$.*

$H^1_{A,\alpha}$ consists of pairs $(a^{0,1}, u) \in \Omega^{0,1}(\mathbf{C}) \oplus \Gamma(E^* \otimes L \otimes K^{1/2})$ satisfying the linearized equation

$$\bar\partial_{AB} u + a^{0,1}\alpha = 0$$

together with the complex Coulomb gauge $(G^c_{A,\alpha})^*(a^{0,1},u) = 0$. By (4.3.7), the tangent space to $\mathfrak{M}^{\mathrm{hol}}$ at consists of triples $(a^{0,1},u,v)$ where $a^{0,1}$, u are as above, $v \in \Gamma(E \otimes L^* \otimes K^{1/2})$, and

$$\begin{cases} \bar\partial_{AB} u + a^{0,1}\alpha = 0, \\ \bar\partial_{AB} v = 0, \\ \alpha v = 0 \end{cases}$$

together with the complex Coulomb gauge for $(a^{0,1},u,v)$. Any non-zero v satisfying the conditions above would give an element (A,α,v) of $\overline{\mathfrak{M}}^{\mathrm{hol}} \backslash \mathfrak{M}^{\mathrm{hol}}$. Since B has been chosen so that $\overline{\mathfrak{M}}^{\mathrm{hol}} \backslash \mathfrak{M}^{\mathrm{hol}}$ is empty, $v = 0$ and the equations obeyed by $(a^{0,1},u,0)$ are identical to the ones defining $H^1_{A,\alpha}$. We conclude that the Zariski tangent spaces to $\mathfrak{N}$ and $\mathfrak{M}^{\mathrm{hol}}$ are equal.

Step 4. *Comparing the orientations.*

Let (A,Ψ) be an irreducible solution of the Seiberg–Witten equations. We have $\Psi = (\alpha, 0)$ where (A,α) is a solution of the framed vortex equations. Consider the deformation operator introduced in Equation 3.2.3:

$$L_{A,\Psi} \colon \Omega^1(i\mathbf{R}) \oplus \Omega^0(i\mathbf{R}) \oplus \Gamma(E^* \otimes S \otimes L) \longrightarrow \Omega^1(i\mathbf{R}) \oplus \Omega^0(i\mathbf{R}) \oplus \Gamma(E^* \otimes S \otimes L)$$

Write $L_{A,\Psi} = L_{A,0} + P$, where

$$L_{A,0} = \begin{pmatrix} *d & -d & 0 \\ -d^* & 0 & 0 \\ 0 & 0 & \not{D}_{AB} \end{pmatrix}$$

and

$$P(a,v,\phi) = (i\,\mathrm{Im}\langle \Psi, \phi\rangle, -2 * \mu(\phi,\Psi), -a\cdot\Psi + v\Psi).$$

The kernel and cokernel of $L_{A,0}$ are naturally identified with

$$H^1(M,i\mathbf{R}) \oplus H^0(M,i\mathbf{R}) \oplus \ker \not{D}_{AB}.$$

The isomorphism between $\det L_{A,0}$ and $\det L_{A,\Psi}$, defining the relative orientation on the obstruction bundle, factors through the determinant space $\det P$ of the finite dimensional map

$$P \colon H^1(M,i\mathbf{R}) \oplus H^0(M,i\mathbf{R}) \oplus \ker \not{D}_{AB} \to H^1(M,i\mathbf{R}) \oplus H^0(M,i\mathbf{R}) \oplus \ker \not{D}_{AB}$$

induced from the zeroth order operator P defined above (for simplicity we use the same letter to denote the induced finite dimensional map). As in the proof of Proposition 4.1.8, we have $H^1(M,i\mathbf{R}) = H^1(S^1,i\mathbf{R}) \oplus H^{0,1}(\Sigma)$. Consider the complex structure on $H^1(S^1,i\mathbf{R}) \oplus H^0(M,i\mathbf{R})$ coming from the identification

$$H^1(S^1,i\mathbf{R}) \oplus H^0(M,i\mathbf{R}) = i\mathbf{R} \oplus i\mathbf{R} = \mathbf{C}.$$

Let $\mathrm{d}t$ be the one-form spanning $H^1(S^1,\mathbf{R})$. Under the Clifford multiplication, $\mathrm{d}t$ acts as the multiplication by i on S, and so $i\mathrm{d}t$ acts as the multiplication by -1. Hene, under the isomorphism $H^1(S^1,i\mathbf{R}) \oplus H^0(M,i\mathbf{R}) = \mathbf{C}$, the map

$(a, v) \mapsto (-a \cdot \Psi + v\Psi)$ is given by $(x + iy) \mapsto (x + iy)\Psi$ and so, in particular, it is complex linear. Next, consider the first two components of P, that is the map $\phi \mapsto (i \operatorname{Im}(\Psi, \phi), -2 * \mu(\phi, \Psi))$. Decompose the moment map into $\mu = \mu_{\mathbf{R}} \oplus \mu_{\mathbf{C}}$ as in the proof of Proposition 4.1.8. The map $\phi \mapsto \mu_{\mathbf{C}}(\phi, \Psi)$ is complex linear from $\ker \not{D}_{AB}$ to $H^{0,1}(\Sigma)$. We are left with the map from $\ker \not{D}_{AB}$ to $H^1(S^1, i\mathbf{R}) \oplus H^0(M, i\mathbf{R})$ given by

$$\phi \mapsto (i \operatorname{Im}(\Psi, \phi) - 2 * \mu_{\mathbf{R}}(\phi, \Psi)).$$

We have $\phi = (u, v)$ under the splitting $S = K^{1/2} \oplus K^{-1/2}$. Following the identifications from the proof of Proposition 4.1.8 we find that $*\mu_{\mathbf{R}}(\phi, \Psi) = -2i \operatorname{Re}(\alpha, u)$ and so our map is

$$\phi = (u, v) \mapsto (i \operatorname{Im}(\alpha, u), -4i \operatorname{Re}(\alpha, u).$$

Up to a constant, it coincides with the complex linear map

$$u \mapsto -\operatorname{Re}(\alpha, u) + i \operatorname{Im}(\alpha, u) = -\overline{(\alpha, u)} = -(u, \alpha).$$

We conclude that the isomorphism $\det P \cong \det L_{A,0}$ agrees with the orientations induced from the complex structures on the cohomology groups. The same is true for $\det P \cong \det L_{A,\Psi}$ where the complex structures on $H^1_{A,\Psi} = \ker L_{A,\Psi}$ and $H^2_{A,\Psi} = \operatorname{coker} L_{A,\Psi}$ come from the isomorphism of analytic spaces $\mathfrak{M} \cong \mathfrak{M}^{\text{hol}}$ given by Theorem 4.3.6. The tangent and obstruction spaces to $\mathfrak{M}^{\text{hol}}$ are canonically identified with the tangent space to $\mathfrak{N}$. Therefore, the relative orientation on the obstruction bundle agrees on the complex orientation on $T^*\mathfrak{N} \to \mathfrak{N}$. □

4.6.3 *Proof of Theorem 4.6.1*

The result follows immediately from combining Proposition 3.2.20, Theorem 4.3.6, and Theorem 4.6.2. For two different generic choices $(\mathbf{p}_0, \eta_0)$ and $(\mathbf{p}_1, \eta_1)$ of parameters in $\mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$, we have

$$n_{\mathfrak{w}}(\mathbf{p}_0, \eta_0) = n_{\mathfrak{w}}(\mathbf{p}_1, \eta_1),$$

because a generic path $(\mathbf{p}_t, \eta_t)_{t \in [0,1]}$ in $\mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$, for which the sign of $d - \tau(\eta_t)$ remains constant, avoids reducibles and harmonic $\mathbf{Z}_2$ spinors, cf. Theorem 4.5.2.

4.7 EXAMPLES AND COMPUTATIONS

In this section we study $\mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta) = \mathfrak{M}^{\text{hol}}_d(\mathbf{p}, \eta)$ using methods of complex geometry. We prove some general properties of the moduli spaces and give their complete description when Σ is a Riemann surface of genus zero, one, or two.

Theorem 4.7.1. *Let Σ be a closed Riemann surface of genus $g(\Sigma)$. Let $M = S^1 \times \Sigma$ be equipped with a spin^c structure $\mathfrak{w}$ induced from a spin structure $K^{1/2} \to \Sigma$ and a Hermitian line bundle $L \to \Sigma$ of degree d. Let $(\mathbf{p}, \eta) \in \mathscr{P}_\Sigma \times \mathscr{Z}_\Sigma$ be a generic choice of S^1–invariant parameters of the Seiberg–Witten equation with two spinors, so that Theorem 4.6.1 holds, and $d - \tau(\eta) < 0$.*

Set $\mathfrak{M} = \mathfrak{M}_{\mathfrak{w}}(\mathbf{p}, \eta)$ and $n = n_{\mathfrak{w}}(\mathbf{p}, \eta)$.

1. *If $d < (1 - g(\Sigma))/2$, then $\mathfrak{M}$ is empty and $n = 0$.*

2. *If $d \geqslant 0$, then $\mathfrak{M}$ admits a holomorphic map to the Jacobian torus of Σ. Its fibers are projective spaces. If $d > 0$, this map is surjective. If $d = 0$, its image is a divisor in the linear system $|2\Theta|$ where Θ is the theta divisor in the Jacobian.*

3. *If $d \geqslant g(\Sigma) - 1$, then $\mathfrak{M}$ is biholomorphic to the projectivisation of a rank $2d$ holomorphic vector bundle over the Jacobian torus of Σ and $n = 0$.*

4. *If $d = 0$ and $g(\Sigma) = 1$, then $\mathfrak{M}$ consists of two points and $n = 2$.*

5. *If $d = 0$ and $g(\Sigma) = 2$, then $\mathfrak{M}$ is biholomorphic to a closed Riemann surface of genus five and $n = 8$.*

In what follows, we will use a complex-geometric description of the moduli spaces $\mathfrak{M}_d^{\mathrm{hol}}(\mathbf{p}, \eta)$ and $\mathfrak{N}_d(\mathbf{p}, \eta)$. Henceforth, we fix a conformal structure on Σ and assume that a perturbing 2–form η is chosen so that the inequality $d - \tau(\eta) < 0$ holds. We will allow the connection $B \in \mathscr{A}(\Sigma, E)$, and thus the holomorphic bundle $\mathscr{E} = \mathscr{E}_B = (E, \bar{\partial}_B)$ to vary. With all this in mind, we will denote the corresponding moduli spaces by $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ and $\mathfrak{N}_d(\mathscr{E})$.

We will say that a statement holds for a generic holomorphic bundle $\mathscr{E}$ if it holds for $\mathscr{E} = (E, \bar{\partial}_B)$ for all B from a residual subset of $\mathscr{A}(\Sigma, E)$.

4.7.1 *Generalized theta divisors*

For $d = 0$, the moduli space $\mathfrak{N}_d(\mathscr{E})$ is related to *generalized theta divisors* [Bea95]. Let $g = g(\Sigma)$ be the genus of Σ and let $\mathrm{Pic}^k(\Sigma)$ denote the component of the Picard group of Σ parametrizing degree k holomorphic line bundles over Σ; for every k it is biholomorphic to the Jacobian torus $\mathrm{Pic}^0(\Sigma)$. Let $\mathscr{E} \to \Sigma$ be a rank 2 stable holomorphic bundle with trivial determinant. For any line bundle $\mathscr{A} \in \mathrm{Pic}^{g-1}(\Sigma)$ the Riemann–Roch theorem gives us

$$\chi(\mathscr{E} \otimes \mathscr{A}) = 2\deg(\mathscr{A}) + 2(1 - g) = 0,$$

so we expect $H^0(\mathscr{E} \otimes \mathscr{A}) = H^1(\mathscr{E} \otimes \mathscr{A}) = 0$ if $\mathscr{A}$ is generic.

Definition 4.7.2. The *generalized theta divisor of* $\mathscr{E}$ is

$$\theta(\mathscr{E}) := \{\mathscr{A} \in \mathrm{Pic}^{g-1}(\Sigma) \mid h^0(\mathscr{E} \otimes \mathscr{A}) > 0\};$$

One can show that $\theta(\mathscr{E})$ is a divisor[3] in $\mathrm{Pic}^{g-1}(\Sigma)$ in the linear system $|2\Theta| = \mathbb{CP}^{2^g-1}$, where Θ is the classical theta divisor

$$\Theta := \{\mathscr{A} \in \mathrm{Pic}^{g-1}(\Sigma) \mid h^0(\mathscr{A}) > 0\}.$$

Proposition 4.7.3. *If $\mathscr{E}$ is a rank 2 stable holomorphic bundle with trivial determinant, then there is a surjective morphism $\mathfrak{N}_0(\mathscr{E}) \to \theta(\mathscr{E})$ whose fibers are projective spaces.*

Proof. A point in $\mathfrak{N}_0(\mathscr{E})$ is an equivalence class $[\mathscr{L}, \alpha]$ where $\mathscr{L} \in \mathrm{Pic}^0(\Sigma)$ and $\alpha \in H^0(\mathscr{E} \otimes \mathscr{L} \otimes K^{1/2})$, with $\alpha \neq 0$. Since $\det(\mathscr{L} \otimes K^{1/2}) = g - 1$, we have $\mathscr{L} \otimes K^{1/2} \in \theta(\mathscr{E})$. The morphism $\mathfrak{N}_0(\mathscr{E}) \to \theta(\mathscr{E})$ is given by $[\mathscr{L}, \alpha] \mapsto \mathscr{L} \otimes K^{1/2}$. The preimage of $\mathscr{L} \otimes K^{1/2}$ is $\mathbb{P}H^0(\mathscr{E} \otimes \mathscr{L} \otimes K^{1/2})$. ☐

3 This is no longer true if $\mathscr{E}$ is of higher rank as it can happen that $\theta(\mathscr{E}) = \mathrm{Pic}^{g-1}(\Sigma)$.

Remark 4.7.4. The divisor $\theta(\mathscr{E})$ can be described geometrically as follows. By a theorem of Lefschetz, the linear system $|2\Theta|$ is base-point free and gives rise to a holomorphic map $\mathrm{Pic}^{g-1}(\Sigma) \to |2\Theta|^*$. It follows that $\theta(\mathscr{E})$, as a subset of $\mathrm{Pic}^{g-1}(\Sigma)$, is the preimage of a hyperplane in $|2\Theta|^*$ under the map $\mathrm{Pic}^{g-1}(\Sigma) \to |2\Theta|^*$. This hyperplane is easy to identify—it is exactly $\theta(\mathscr{E})$ thought of as a point in $|2\Theta|$ or, equivalently, as a hyperplane in $|2\Theta|^*$. Varying the background bundle $\mathscr{E}$, we vary the corresponding hyperplane and therefore the divisor $\theta(\mathscr{E}) \subset \mathrm{Pic}^{g-1}(\Sigma)$.

4.7.2 *General properties of the moduli spaces*

Proposition 4.7.5. *For a generic choice of $\mathscr{E}$ the following holds.*

1. *If $d < \frac{1-g}{2}$, then $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is empty.*
2. *If $d \geqslant 0$, then $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is non-empty.*
3. *If $d \geqslant g-1$, then $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is Zariski smooth with the underying complex manifold biholomorphic to the projectivisation of a rank $2d$ vector bundle over $\mathrm{Pic}^d(\Sigma)$.*

Remark 4.7.6. Proposition 4.7.5 shows that the most interesting case is $(1-g)/2 \leqslant d < 0$. It is an interesting question whether $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is generically non-empty for d in this range.

The proof of Proposition 4.7.5 relies on the following general result about holomorphic vector bundles on Riemann surfaces. Recall that $\mathscr{E}$ *stable* if for any holomorphic line bundle $\mathscr{A}$ the existence of a non-zero holomorphic map $\mathscr{A} \to \mathscr{E}$ implies $\deg(\mathscr{A}) < 0$.

Lemma 4.7.7. *If $g \geqslant 2$, then there is an open dense subset of $\mathscr{A}(\Sigma, E)$ such that for every connection B from this subset the corresponding holomorphic bundle $\mathscr{E}_B$ is stable.*

Proof. $\mathscr{E}_B$ fails to be stable if and only if there is a holomorphic line bundle $\mathscr{L}$ with $\deg(\mathscr{L}) = d \geqslant 0$ and a non-zero map $\theta\colon \mathscr{L} \to \mathscr{E}_B$. In other words, if L is a unitary bundle underlying $\mathscr{L}$ and A is a connection inducing $\mathscr{L}$, then $\theta \in \Gamma(L^* \otimes E)$ satisfies $\bar\partial_{AB}\theta = 0$. Consider

$$\mathcal{U}_d := \{B \in \mathscr{A}(\Sigma, E) \mid \ker \bar\partial_{AB} = \{0\} \text{ for all } A \in \mathscr{A}(\Sigma, L)\}$$

for a fixed degree d unitary bundle L.

Step 1. *$\mathcal{U}_d$ is open.*

Let $B \in \mathcal{U}_d$. For every $A \in \mathscr{A}(\Sigma, L)$ there is a neighbourhood of (A, B) in $\mathscr{A}(\Sigma, L) \times \mathscr{A}(\Sigma, E)$ such that for all (A', B') from this neighbourhood $\ker \bar\partial_{A'B'} = 0$. Since this condition is invariant under the action of $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$, and

$$A(\Sigma, L)/\mathscr{G}^{\mathbf{C}}(\Sigma, L) = \mathrm{Pic}^d(\Sigma)$$

is compact, there is a neighbourhood of B in $\mathscr{A}(\Sigma, E)$ such that for all B' from this neighbourhood $\ker \bar\partial_{AB'} = 0$ for all A. All such B' belong to $\mathcal{U}_d$ which proves that $\mathcal{U}_d$ is open.

Step 2. *$\mathcal{U}_d$ is dense.*

The proof is similar to that of Proposition 3.2.6. In what follows we replace the spaces of connections and sections by their Sobolev completions. Cover $\mathscr{A}(\Sigma, L)/\mathscr{G}^{\mathbf{C}}(\Sigma, L)$ by finitely many charts that can be lifted to $\mathscr{G}^{\mathbf{C}}(\Sigma, L)$–slices in $\mathscr{A}(\Sigma, L)$. Let V be such a chart; it is an open subset of $\mathbf{R}^{2g}$ parametrizing a smooth family of connections $\{A_x\}_{x \in U}$. The claim will follow if can show that

$$\mathcal{U}_V := \{B \in \mathscr{A}(\Sigma, E) \mid \ker \bar{\partial}_{A_x B} = \{0\} \text{ for all } x \in V\}$$

is dense in $\mathscr{A}(\Sigma, E)$. To prove this, consider

$$S := \{\theta \in \Gamma(\Sigma, L^* \otimes E) \mid \|\theta\|_{L^2} = 1\}$$

and the map

$$f \colon \mathscr{A}(\Sigma, E) \times V \times S \longrightarrow \Omega^{0,1}(\Sigma, L^* \otimes E)$$

$$f(B, x, \theta) := \bar{\partial}_{A_x B} \theta.$$

For every $B \in \mathscr{A}(\Sigma, E)$ the restriction $f_B := f(B, \cdot, \cdot)$ is a Fredholm map (between suitable Sobolev spaces) because its derivative is the sum of $\bar{\partial}_{A_x B}$ and the derivative with respect to x, which is a finite-dimensional operator. By the Riemann–Roch theorem,

$$\begin{aligned} \operatorname{ind}_{\mathbf{R}} df_B &= \dim V + 2 \operatorname{ind} \bar{\partial}_{A_x B} - 1 \\ &= 2g + 4(-d + 1 - g) - 1 \\ &= 2(-2d + 2 - g) - 1 \leqslant 0, \end{aligned} \tag{4.7.1}$$

where we subtract 1 because $\bar{\partial}_{A_x B}$ is restricted to the tangent space to S. A computation similar to that in the proof of Proposition 3.2.6 shows that the derivative of the full map f is surjective at every point of $f^{-1}(0)$. By the Sard–Smale theorem, the set $f_B^{-1}(0)$ is empty for B from a dense subset of $\mathscr{A}(\Sigma, E)$; all such B belong to $\mathcal{U}_V$.

Step 3. *$\mathcal{U} := \bigcap_{d \geqslant 0} \mathcal{U}_d$ is open and dense.*

$\mathcal{U}$ is dense by Baire's theorem; it is open by the following argument. By (4.4.3), the existence of a destabilising map $\theta \colon \mathscr{L} \to \mathscr{E}_B$ implies

$$0 \leqslant d \leqslant h^0(\Sigma, \mathscr{E}_B) + 2g - 2.$$

The right-hand side can only decrease when B is replaced by a sufficiently close B'. (Indeed, if we split $\Gamma(\Sigma, E)$ into $\ker \bar{\partial}_B$ and its L^2–orthogonal complement Q, then by the elliptic estimate $\bar{\partial}_{B'}$ is non-degenerate when restricted to Q for all nearby connections B'; it follows that the projection $\ker \bar{\partial}_{B'} \to \ker \bar{\partial}_B$ is injective. See also [Muk03, Proposition 11.21] for an algebro-geometric proof.) Therefore, to guarantee that a nearby connection B' belongs to $\mathcal{U}$ it is enough to check that it belongs to $\mathcal{U}_d$ for finitely many values of d. Thus, for every $B \in \mathcal{U}$ there are finitely many open neighbourhoods of B whose intersection lies entirely in $\mathcal{U}$. □

Proof of Proposition 4.7.5. By Theorem 4.6.2, for a generic choice of $\mathscr{E}$ the moduli space $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E}) = \mathfrak{N}_d(\mathscr{E})$ is a compact complex manifold of dimension $g - 1 + 2d$. If $d < (1 - g)/2$, then this dimension is negative and $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ must be empty.

The case $d = 0$ was discussed in the previous subsection. Let $\deg(\mathscr{L}) = d > 0$; then

$$h^0(\mathscr{E} \otimes \mathscr{L} \otimes K^{1/2}) - h^1(\mathscr{E} \otimes \mathscr{L} \otimes K^{1/2}) = 2d > 0;$$

thus, $\mathscr{L}$ is in the image of the projection $\pi\colon \mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E}) \to \mathrm{Pic}^d(\Sigma)$ given by $[\mathscr{L}, \alpha, \beta] \mapsto \mathscr{L}$. For a generic $\mathscr{E}$, by Theorem 4.6.2, $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E}) = \mathfrak{N}_d(\mathscr{E})$ and so $\pi^{-1}(\mathscr{L}) = \mathbf{P}H^0(\mathscr{E} \otimes \mathscr{L})$.

For the proof of the third item, assume $g \geqslant 2$; the cases $g = 0, 1$ will be considered separately in the next section. By Lemma 4.7.7, a generic $\mathscr{E}$ is stable and by Serre duality,

$$h^1(\mathscr{E} \otimes \mathscr{L} \otimes K^{1/2}) = h^0(\mathscr{E}^* \otimes \mathscr{L}^* \otimes K^{1/2}).$$

Any element of $H^0(\mathscr{E}^* \otimes \mathscr{L}^* \otimes K^{1/2})$ gives a holomorphic map $\mathscr{L} \otimes K^{-1/2} \to \mathscr{E}^*$. We have

$$\deg(\mathscr{L} \otimes K^{-1/2}) = d - g + 1 \geqslant 0.$$

Since $\mathscr{E}^*$ is stable, it follows that any holomorphic map $\mathscr{L} \otimes K^{-1/2} \to \mathscr{E}^*$ is trivial. Thus, $h^1(\mathscr{E} \otimes \mathscr{L} \otimes K^{1/2}) = 0$ and by the Riemann–Roch theorem $h^0(\mathscr{E} \otimes \mathscr{L} \otimes K^{1/2}) = 2d$ for every $\mathscr{L} \in \mathrm{Pic}^d(\Sigma)$. We conclude that $\pi\colon \mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E}) \to \mathrm{Pic}^d(\Sigma)$ is the projectivisation of the rank $2d$ vector bundle whose fiber over $\mathscr{L}$ is the cohomology group $H^0(\mathscr{E} \otimes \mathscr{L} \otimes K^{1/2}) = \mathbf{C}^{2d}$. □

4.7.3 *Genus zero*

Let $\Sigma = \mathbb{CP}^1$. For $k \in \mathbb{Z}$ denote by $\mathcal{O}(k)$ the unique holomorphic line bundle of degree k; $K^{1/2} = \mathcal{O}(-1)$ is the unique spin structure. By a theorem of Grothendieck, every holomorphic bundle over $\mathbb{CP}^1$ is the direct sum of line bundles. In particular, every holomorphic $\mathrm{SL}(2, \mathbf{C})$–bundle is of the form $\mathscr{E} = \mathcal{O}(k) \oplus \mathcal{O}(-k)$ for some $k \geqslant 0$, with $k = 0$ being the generic case.

Proposition 4.7.8. *Let $\Sigma = \mathbb{CP}^1$ and $\mathscr{E} = \mathcal{O}(k) \oplus \mathcal{O}(-k)$ for $k \geqslant 0$.*

1. *If $d \leqslant 0$ and $k \leqslant |d|$, then $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is empty.*
2. *If $d > 0$ and $k \leqslant d$, then $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is Zariski smooth with the underlying complex manifold biholomorphic to $\mathbb{CP}^{2d-1}$.*
3. *If $k > |d|$, then $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is non-compact and its compactification $\overline{\mathfrak{M}}_d^{\mathrm{hol}}(\mathscr{E})$ is homeomorphic to a locally trivial $\mathbb{CP}^{k-d}$–fibration over $\mathbb{CP}^{k+d}$.*

Proof. $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ consists of the equivalences classes of pairs (α, β) such that

$$\alpha \in H^0(\mathcal{O}(k + d - 1)) \oplus H^0(\mathcal{O}(-k + d - 1)),$$

$$\beta \in H^0(\mathcal{O}(-k - d - 1)) \oplus H^0(\mathcal{O}(k - d - 1)),$$

$\alpha \neq 0$, and $\alpha\beta = 0 \in H^0(\mathcal{O}(-2))$—this is automatically satisfied since $h^0(\mathcal{O}(-2)) = 0$. If $\mathscr{E}$ is generic, so that $k = 0$, then $d \leqslant 0$ which implies that $\alpha = 0$ and $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is empty. If $d > 0$, then $\alpha \in \mathbb{C}^{2d}$ and $\beta = 0$; it follows that

$$\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E}) = \mathfrak{N}_d(\mathscr{E}) = \mathbb{CP}^{2d-1}.$$

The same description of $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is valid in the non-generic case $k \neq 0$ as long as $k \leqslant |d|$. When $k > |d|$ the moduli space is no longer compact and harmonic $\mathbf{Z}_2$ spinors appear. If $k > d > 0$, then $\alpha \in \mathbf{C}^{k+d}$, $\beta \in \mathbf{C}^{k-d}$ and $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is the total space of the vector bundle $\mathcal{O}(-1)^{\oplus(k-d)}$ over $\mathbb{CP}^{k+d-1}$. The compactification $\overline{\mathfrak{M}}_d^{\mathrm{hol}}(\mathscr{E})$ is the $\mathbb{CP}^{k-d}$–bundle over $\mathbb{CP}^{k+d-1}$ obtained from the projectivisation of the vector bundle $\mathcal{O}(-1)^{\oplus(k-d)} \oplus \mathcal{O}$. □

4.7.4 *Genus one*

Let $\Sigma = S^1 \times S^1$ equipped with a complex structure making it into an elliptic curve. Isomorphism classes of line bundles of a given degree d on Σ form the Jacobian $\mathrm{Pic}^d(\Sigma)$ which is isomorphic to the dual torus Σ^*. The canonical bundle of Σ is trivial and without loss of generality we can take $K^{1/2}$ also to be trivial. Holomorphic vector bundles over elliptic curves were classified by Atiyah [Ati57]. A generic holomorphic $SL(2,\mathbf{C})$–bundle $\mathscr{E}$ is of the form $\mathscr{E} = \mathscr{A} \oplus \mathscr{A}^{-1}$ for a degree zero line bundle $\mathscr{A} \to \Sigma$. We may moreover assume that $\mathscr{A}^2 \neq \mathcal{O}$ since there are only four line bundles satisfying $\mathscr{A}^2 = \mathcal{O}$.

Proposition 4.7.9. *Let Σ be an elliptic curve. Suppose that $\mathscr{E}$ is generic, that is of the form $\mathscr{E} = \mathscr{A} \oplus \mathscr{A}^{-1}$ for $\mathscr{A} \in \mathrm{Pic}^0(\Sigma)$ satisfying $\mathscr{A}^2 \neq \mathcal{O}$.*

1. *If $d < 0$, then $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is empty.*
2. *If $d > 0$, then $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is Zariski smooth with the underlying complex manifold biholomorphic to the projectivisation of a rank $2d$ vector bundle over $\mathrm{Pic}^d(\Sigma)$.*
3. *If $d = 0$, then $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is regular and consists of two points.*

Remark 4.7.10. If $d > 0$, then the cohomology ring $H^*(\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E}), \mathbf{Z})$ is isomorphic as $H^*(\mathrm{Pic}^d(\Sigma), \mathbf{Z})$–modules to $H^*(\mathrm{Pic}^d(\Sigma), \mathbf{Z})[H]/(H^{2d})$ where $\deg(H) = 2$. In particular,

$$\chi(\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})) = \chi(\mathrm{Pic}^d(\Sigma))\chi(\mathbb{CP}^{2d-1}) = 0.$$

Proof of Proposition 4.7.9. $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ consists of equivalence classes of triples $(\mathscr{L}, \alpha, \beta)$ where $\mathscr{L} \in \mathrm{Pic}^d(\Sigma)$ and

$$\alpha \in H^0(\mathscr{A} \otimes \mathscr{L}) \oplus H^0(\mathscr{A}^{-1} \otimes \mathscr{L}),$$

$$\beta \in H^0(\mathscr{A}^{-1} \otimes \mathscr{L}^{-1}) \oplus H^0(\mathscr{A} \otimes \mathscr{L}^{-1}),$$

satisfying $\alpha \neq 0$ and $\alpha\beta = 0$ in $H^0(\mathcal{O}) = \mathbf{C}$.

For $d < 0$ we must have $\alpha = 0$ and so the moduli space is empty. For $d = 0$ the only choices of $\mathscr{L}$ for which α is possibly non-zero are $\mathscr{L} = \mathscr{A}^{-1}$ and $\mathscr{L} = \mathscr{A}$. If $\mathscr{L} = \mathscr{A}^{-1}$, then

$$\alpha \in H^0(\mathcal{O}) \oplus H^0(\mathscr{A}^{-2}),$$

$$\beta \in H^0(\mathcal{O}) \oplus H^0(\mathscr{A}^2).$$

Since $\mathscr{A}^2$ and $\mathscr{A}^{-2}$ are non-trivial, so they have no non-zero sections. The only possibly choice for α, up to scaling, is therefore $\alpha = (1,0)$ and the condition $\alpha\beta = 0$ forces β to be zero since the pairing $H^0(\mathcal{O}) \times H^0(\mathcal{O}) \to H^0(\mathcal{O})$ is simply the multiplication $\mathbf{C}\times\mathbf{C}\to\mathbf{C}$. We repeat the same argument for $\mathscr{L}=\mathscr{A}$ and conlude that $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ consists of two isolated points. In particular it is compact and so Zariski smooth thanks to Corollary 4.6.3. Since it is also has the correct dimension zero, we conclude that each of the points is regular.

Consider now the case $d>0$. For every $\mathscr{L}\in\mathrm{Pic}^d(\Sigma)$ the Riemann–Roch theorem gives us

$$h^0(\mathscr{L}\otimes\mathscr{A})-h^1(\mathscr{L}\otimes\mathscr{A})=d$$

and by Serre duality, $h^1(\mathscr{L}\otimes\mathscr{A})=h^0(\mathscr{L}^{-1}\otimes\mathscr{A})=0$ because $\deg(\mathscr{L}^{-1})<0$, Thus $H^0(\mathscr{L}\otimes\mathscr{A})=\mathbf{C}^d$ and the same is true if $\mathscr{A}$ is replaced by $\mathscr{A}^{-1}$. Therefore, α is identified with a non-zero vector in $\mathbf{C}^{2d}$. On the other hand, $\beta=0$ since $\mathscr{A}^{\pm1}\otimes\mathscr{L}^{-1}$ has no non-trivial sections. Since the above discussion is valid for any $\mathscr{L}\in\mathrm{Pic}^d(\Sigma)$, it follows that $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is a locally trivial $\mathbb{CP}^{2d-1}$–fibration over $\mathrm{Pic}^d(\Sigma)$: the projectivisation of a rank $2d$ holomorphic vector bundle over $\mathrm{Pic}^d(\Sigma)$ given by the push-forward of the Poincaré line bundle $\mathcal{P}\to\mathrm{Pic}^d(\Sigma)\times\Sigma$ to the first factor. □

It is worthwhile discussing some non-generic examples. The cases when $\mathscr{E}=\mathscr{A}\oplus\mathscr{A}^{-1}$ and either $\deg(\mathscr{A})\neq0$ or $\mathscr{A}^2=\mathcal{O}$ are similar to the ones already considered. Another possibility is that $\mathscr{E}$ is indecomposable, in which case it is of the form $\mathscr{E}=\mathscr{E}_0\otimes\mathscr{A}$ where $\mathscr{A}\in\mathrm{Pic}^0(\Sigma)$ satisfies $\mathscr{A}^2=\mathcal{O}$ and $\mathscr{E}_0$ is the unique non-trivial bundle obtained as an extension

$$0\longrightarrow\mathcal{O}\longrightarrow\mathscr{E}_0\longrightarrow\mathcal{O}\longrightarrow0.$$

The line bundle $\mathscr{A}$ is uniquely determined by $\mathscr{E}$.

Example 4.7.11. Suppose without loss of generality that $\mathscr{E}=\mathscr{E}_0$. It is shown in [Ati57] that if $h^0(\mathscr{E}\otimes\mathscr{L})\neq0$, then either $\deg(\mathscr{L})>0$ or $\mathscr{L}=\mathcal{O}$ in which case we have $h^0(\mathscr{E})=1$. We conclude that when $d<0$ or $d>0$ the moduli space $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is, respectively, empty or the projectivisation of a vector bundle over $\mathrm{Pic}^d(\Sigma)$. On the other hand, for $d=0$ the only choice of $\mathscr{L}$ for which $h^0(\mathscr{E}\otimes\mathscr{L})>0$ is $\mathscr{L}=\mathcal{O}$ and we look for holomorphic sections

$$\alpha\in H^0(\mathscr{E})=\mathbf{C},$$

$$\beta\in H^0(\mathscr{E}^*)=\mathbf{C}$$

such that $\alpha\neq0$ and $\alpha\beta=0$. Up to scaling, $\alpha=1$. We will show now that the pairing $H^0(\mathscr{E})\times H^0(\mathscr{E}^*)\to\mathbf{C}$ is trivial and, as a consequence, β can be chosen arbitrarily. Let $\Omega\in H^0(\Lambda^2\mathscr{E})$ be a nowhere vanishing holomorphic volume form. It induces an isomorphism $\mathscr{E}\to\mathscr{E}^*$ given by $v\mapsto\Omega(v,\cdot)$. If α is a generator of $H^0(\mathscr{E})$, then $\gamma=\Omega(\alpha,\cdot)$ is a non-zero holomorphic section of $H^0(\mathscr{E}^*)$ and so it must be a generator. On the other hand, $\gamma(\alpha)=\Omega(\alpha,\alpha)=0$ since Ω is skew-symmetric—this shows that $\alpha\beta=0$ for every $\beta\in H^0(\mathscr{E}^*)$. Therefore, $\mathfrak{M}_0^{\mathrm{hol}}(\mathscr{E})$ is homeomorphic to $\mathbf{C}$. Its compactification $\overline{\mathfrak{M}}_0^{\mathrm{hol}}(\mathscr{E})$ is homeomorphic to $\mathbb{CP}^1$.

4.7.5 *Genus two*

Let Σ be a genus two Riemann surface. By Lemma 4.7.7, a generic holomorphic bundle on Σ is stable. The proof of the next lemma can be found in [NR69].

Lemma 4.7.12. *Let $\mathscr{W}$ be a stable rank two vector bundle with trivial determinant over a genus two closed Riemann surface Σ. If $\mathscr{A} \to \Sigma$ is a degree 1 line bundle, then*

1. $h^0(\mathscr{W} \otimes \mathscr{A}) \leqslant 1$.
2. *Any non-zero homomorphism $\mathscr{A}^* \to \mathscr{W}$ is everywhere injective.*

Proposition 4.7.13. *Let Σ be a closed Riemann surface of genus two. For a generic holomorphic $\mathrm{SL}(2, \mathbf{C})$–bundle $\mathscr{E} \to \Sigma$ we have the following description of $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$.*

1. *If $d < 0$, then $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is empty.*
2. *If $d = 0$, then $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is Zariski smooth with the underlying complex manifold biholomorphic to a closed Riemann surface of genus five.*
3. *If $d > 0$, then $\mathfrak{M}_d^{\mathrm{hol}}(\mathscr{E})$ is Zariski smooth with the underlying complex manifold biholomorphic to the projectivisation of a rank $2d$ vector bundle over $\mathrm{Pic}^d(\Sigma)$.*

Proof. Items (2) and (3) follow from Proposition 4.7.5 For $d = 0$ we use the relation between the moduli space of framed vortices and theta divisors described in Section 4.7.1. Let $\mathscr{E} \to \Sigma$ be a stable $\mathrm{SL}(2, \mathbf{C})$–bundle. Let $\mathcal{SU}(2)$ be the compactification of the moduli space of such bundles obtained by adding the S–equivalence classes of semi-stable bundles. As explained in [NR69], we have $|2\Theta| = \mathbb{CP}^3$ and the map introduced in Section 4.7.1

$$\theta\colon \mathcal{SU}(2) \to \mathbb{CP}^3$$

$$\mathscr{E} \mapsto \theta(\mathscr{E})$$

is an isomorphism. Recall that $\theta(\mathscr{E})$ can be seen either as a subset of $\mathrm{Pic}^1(\Sigma)$

$$\theta(\mathscr{E}) = \{A \in \mathrm{Pic}^1(\Sigma) \mid h^0(\mathscr{E} \otimes A) > 0\}$$

or as the corresponding point in $|2\Theta|$.

$\mathrm{Pic}^1(\Sigma)$ is a 2–dimensional complex torus and the map $\mathrm{Pic}^1(\Sigma) \to |2\Theta|^* = (\mathbb{CP}^3)^*$ induces a degree four embedding of the Kummer surface $\mathrm{Pic}^1(\Sigma)/\mathbf{Z}_2$. Thus, as a subset $\theta(\mathscr{E}) \subset \mathrm{Pic}^1(\Sigma)$ is the preimage of the intersection of the Kummer surface in $(\mathbb{CP}^3)^*$ with the hyperplane $\theta(\mathscr{E}) \in \mathbb{CP}^3$ under the quotient map $\mathrm{Pic}^1(\Sigma) \to \mathrm{Pic}^1(\Sigma)/\mathbf{Z}^2$. Since $\theta\colon \mathcal{SU}(2) \to \mathbb{CP}^3$ is an isomorphism, by changing the background bundle $\mathscr{E}$ we can obtain all hyperplanes in $(\mathbb{CP}^3)^*$. In particular, for a generic choice of $\mathscr{E}$, the hyperplane $\theta(\mathscr{E})$ will avoid all the 16 singular points of $\mathrm{Pic}^1(\Sigma)/\mathbf{Z}_2$ and the intersection $\mathrm{Pic}^1(\Sigma)/\mathbf{Z}_2 \cap \theta(\mathscr{E})$ will be a smooth complex curve of degree four and genus three. Its preimage under $\mathrm{Pic}^1(\Sigma) \to \mathrm{Pic}^1(\Sigma)/\mathbf{Z}_2$ is a smooth curve $C \subset \mathrm{Pic}^1(\Sigma)$ of genus five, by the Hurwitz formula.

Let $\pi\colon \mathfrak{M}_0^{\mathrm{hol}}(\mathscr{E}) \to C$ be the composition of $\mathfrak{M}_0^{\mathrm{hol}}(\mathscr{E}) \to \mathfrak{N}_0(\mathscr{E})$ with the projection $\mathfrak{N}_0(\mathscr{E}) \to \theta(\mathscr{E}) = C$ from Section 4.7.1. We claim that this map is an isomorphism for a generic choice of $\mathscr{E}$. In order to prove that, it is enough to check that the fiber over any line bundle $\mathscr{L} \otimes K^{1/2}$ in C consists of one point. This is equivalent to showing that $H^0(\mathscr{E} \otimes \mathscr{L} \otimes K^{1/2})$ is spanned by a single non-zero section α and if $\beta \in H^0(\mathscr{E}^* \otimes \mathscr{L}^* \otimes K^{1/2})$ is any section satisfying $\alpha\beta = 0$, then $\beta = 0$. The first claim follows immediately from Lemma 4.7.12. As regards the second claim, assume that α and β are as above and $\beta \neq 0$. By Lemma 4.7.12, the homomorphisms

$$\alpha\colon \mathscr{L}^* \otimes K^{-1/2} \longrightarrow \mathscr{E} \quad \text{and} \quad \beta\colon \mathscr{L} \otimes K^{-1/2} \longrightarrow \mathscr{E}^*$$

are everywhere injective. Since $\operatorname{rank} \mathscr{E} = 2$, $\alpha\beta = 0$ implies the exactness of the sequence

$$0 \longrightarrow \mathscr{L}^* \otimes K^{-1/2} \xrightarrow{\alpha} \mathscr{E} \xrightarrow{\beta^t} \mathscr{L}^* \otimes K^{1/2} \longrightarrow 0. \tag{4.7.2}$$

Since $\det \mathscr{E} = \mathcal{O}$, we conclude $\mathscr{L}^2 = \mathcal{O}$. There are 16 line bundles satisfying this condition: order two elements of $\mathrm{Pic}^0(\Sigma)$. For each of them, all possible non-trivial extensions $\mathscr{E}$ as above are classified by the corresponding extension class in $\mathbb{P}H^1(K^{-1}) = \mathbb{CP}^2$. (Note that the extension is non-trivial because $\mathscr{E}$ is stable.) Thus, all stable bundles $\mathscr{E}$ that can be represented as such an extension form a proper subvariety of $\mathcal{SU}(2) = \mathbb{CP}^3$ consisting of the images of 16 maps $\mathbb{CP}^2 \to \mathcal{SU}(2)$. A generic stable bundle $\mathscr{E}$ will not belong to this subvariety. In this case, we conclude that each fiber of the map π consists of a single point and π is an isomorphism. In particular, $\mathfrak{M}_0^{\mathrm{hol}}(\mathscr{E})$ is compact and therefore Zariski smooth by Corollary 4.6.3. □

Remark 4.7.14. Note that the last part of the proof was unnecessary; we already know that generically $\mathfrak{M}_0^{\mathrm{hol}}(\mathscr{E}) = \mathfrak{N}_0(\mathscr{E})$ is compact and Zariski smooth which is enough to conclude $\mathfrak{M}_0^{\mathrm{hol}}(\mathscr{E}) = C$. On the other hand, the argument presented above identifies the locus of those semi-stable bundles $\mathscr{E} \in \mathcal{SU}(2)$ for which harmonic $\mathbf{Z}_2$ spinors appear. It consists of strictly semi-stable bundles $\mathscr{A} \oplus \mathscr{A}^{-1}$, for some $\mathscr{A} \in \mathrm{Pic}^0(\Sigma)$, which form the Kummer surface $\mathrm{Pic}^1(\Sigma)/\mathbf{Z}_2$ in $\mathcal{SU}(2) = \mathbb{CP}^3$, and stable bundles $\mathscr{E}$ that arise from an extension of the form (4.7.2) for some element $\mathscr{L} \in \mathrm{Pic}^0(\Sigma)$ of order two. The latter form a subvariety covered by the images of 16 maps $\mathbb{CP}^2 \to \mathbb{CP}^3$.

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