ANALYSIS AND OPTIMIZATION, PRACTICE PROBLEMS

Problem 1. (1) Suppose we want to minimize \( \int_0^1 x^2(t) - (x'(t))^2 \, dt \) over all \( C^1 \) functions \( x(t) \) such that \( x(0) = 0 \) and \( x(1) = 1 \). Write down a differential equation satisfied by such a minimizer.

(2) Suppose we want to minimize \( \int_1^2 \frac{(x'(t))^2}{t^2} \, dt \) subject to \( x(1) = 0 \) and \( x(2) = 1 \). Write down the differential equation satisfied by the minimizer. Given that the solution is of the form \( at^3 + bt^2 + ct + d \), find the solution to this differential equation (with the condition on endpoints).

Problem 2. Do Larson 9.3 problem 10. At each step of the simplex algorithm, keep track of which constraints are active and which are inactive, and label your pivot entry. What is the dual minimization problem?

Problem 3. Maximize the function \( f(x, y) = xy + x^2 \) subject to constraints \( x^2 + y \leq 2 \) and \( y \geq 0 \). Write down the Kuhn-Tucker conditions; solve them; justify why your answer is a maximum. Say the phrase “complementary slackness” five times quickly and remember what it means.

Problem 4. Suppose we are minimizing a function \( f(x_1, \ldots, x_n) \) under the conditions \( g_1(x_1, \ldots, x_n) = g_2(x_1, \ldots, x_n) = \cdots = g_r(x_1, \ldots, x_n) = 0 \). Under what hypotheses is a solution to the Lagrangian multiplier equations automatically a global minimum?

Problem 5. (1) Write down examples of open sets, closed sets, bounded sets, and compact sets. Review the notion of sequential compactness. Write down non-examples of all of these too.

(2) Let \( f(x) \) be a convex function on \( \mathbb{R}^n \). Given two points \( x \) and \( y \) what is the relation between \( \nabla f(x) \), \( y - x \), and \( f(y) - f(x) \)?

Problem 6. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a strictly convex function such that \( f(x) \geq \|x\|^2 \) for all \( x \). Show that every possible value of the gradient is realized exactly once: that is, for each \( p \in \mathbb{R}^n \), there is exactly one point \( x = x(p) \) such that \( \nabla f(x(p)) = p^\top \). Hint: Optimize the function \( g(x) = f(x) - p \cdot x \).