HW 7, due 3/28:

4.3: 62; 5.1: 10, 16, 26, 42, 44
5.2: 4, 34, 36; 5.3: 4, 8, 26, 36

4.3.62 a Finding this basis is equivalent to finding a basis of the kernel of \( \begin{bmatrix} 1 & -2 & 2 \end{bmatrix} \) that does not contain any zeroes. We can quickly spot the vectors \( \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \), so \( B = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 4 \\ 1 \\ -1 \end{pmatrix} \), for example.

b From Theorem 4.3.3, \( S_{B \to A} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 4 \\ 1 \\ -1 \end{pmatrix} \).

c \( S_{A \to B} = S_{B \to A}^{-1} = \frac{1}{3} \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} \).

d Theorem 4.3.4 reveals that \( [\vec{b}_1 \vec{b}_2] = [\vec{a}_1 \vec{a}_2] S_{B \to A} \).

5.1.10 \( \vec{a} \cdot \vec{b} = 2 + 3k + 4 = 6 + 3k \). The two vectors enclose a right angle if \( \vec{a} \cdot \vec{b} = 0 + 3k = 0 \), that is, if \( k = -2 \).

5.1.16 You may be able to find the solutions by educated guessing. Here is the systematic approach: we first find all vectors \( \vec{x} \) that are orthogonal to \( \vec{v}_1, \vec{v}_2, \text{ and } \vec{v}_3 \), then we identify the unit vectors among them.

Finding the vectors \( \vec{x} \) with \( \vec{x} \cdot \vec{v}_1 = \vec{x} \cdot \vec{v}_2 = \vec{x} \cdot \vec{v}_3 = 0 \) amounts to solving the system
\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 &= 0 \\
x_1 + x_2 - x_3 - x_4 &= 0 \\
x_1 - x_2 + x_3 - x_4 &= 0
\end{align*}
\]

(we can omit all the coefficients \( \frac{1}{2} \)).

The solutions are of the form \( \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -t \\ -t \\ -t \\ t \end{pmatrix} \).

Since \( \|\vec{x}\| = 2|t| \), we have a unit vector if \( t = \frac{1}{2} \) or \( t = -\frac{1}{2} \). Thus there are two possible choices for \( \vec{v}_4 \):

\[
\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}.
\]

5.1.26 The two given vectors spanning the subspace are orthogonal, but they are not unit vectors; both have length 7. To obtain an orthonormal basis \( \vec{u}_1, \vec{u}_2 \) of the subspace, we divide by 7:

\[
\vec{u}_1 = \frac{1}{7} \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}, \vec{u}_2 = \frac{1}{7} \begin{pmatrix} -3 \\ -6 \\ 2 \end{pmatrix}.
\]
Now we can use Theorem 5.1.5, with $x = \begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$:

$$\text{proj}_V \bar{x} = (\bar{u}_1 \cdot \bar{x}) \bar{u}_1 + (\bar{u}_2 \cdot \bar{x}) \bar{u}_2 = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 19 \\ 39 \\ 64 \end{bmatrix}.$$ 

5.1.42 $||\bar{v}_1 + \bar{v}_2|| = \sqrt{(\bar{v}_1 + \bar{v}_2) \cdot (\bar{v}_1 + \bar{v}_2)} = \sqrt{a_{11} + 2a_{12} + a_{22}} = \sqrt{22}$.

5.1.44 One method to solve this is to take $\bar{v} = \bar{v}_2 - \text{proj}_{\bar{v}_3} \bar{v}_2 = \bar{v}_2 - \frac{20}{49} \bar{v}_3$.

5.2.4 $\bar{u}_1 = \frac{1}{5} \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ and $\bar{u}_2 = \frac{1}{5} \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}$ as in Exercise 3.

Since $\bar{v}_3$ is orthogonal to $\bar{u}_1$ and $\bar{u}_2$, $\bar{u}_3 = \frac{1}{||\bar{v}_3||} \bar{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

5.2.34 $\text{ref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

A basis of $\ker(A)$ is $\bar{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$, $\bar{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$.

We apply the Gram-Schmidt process and obtain

$$\bar{u}_1 = \frac{1}{||\bar{v}_1||} \bar{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{u}_2 = \frac{\bar{v}_2 - (\bar{v}_2 \cdot \bar{u}_1) \bar{u}_1}{||\bar{v}_2 - (\bar{v}_2 \cdot \bar{u}_1) \bar{u}_1||} = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}.$$
5.2.36 Write \( M = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & -4 & 6 \\ 0 & 0 & 7 \end{bmatrix} \)
\[ \uparrow \quad Q_0 \quad \uparrow \quad R_0 \]

This is almost the QR factorization of \( M \): the matrix \( Q_0 \) has orthonormal columns and \( R_0 \) is upper triangular; the only problem is the entry \(-4\) on the diagonal of \( R_0 \). Keeping in mind how matrices are multiplied, we can change all the signs in the second column of \( Q_0 \) and in the second row of \( R_0 \) to fix this problem:

\[
M = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix}
\]
\[ \uparrow \quad Q \quad \uparrow \quad R \]

5.3.4 Not orthogonal, the first and third column vectors fail to be perpendicular to each other.

5.3.8 \( A + B \) will not necessarily be orthogonal, because the columns may not be unit vectors. For example, if \( A = B = I_n \), then \( A + B = 2I_n \), which is not orthogonal.


5.3.36 Let the third column be the cross product of the first two: \( A = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{18}} \\ \frac{2}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} \\ \frac{1}{3} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} \end{bmatrix} \).

There is another solution, with the signs in the last column reversed.