MATH V2010 Linear Algebra
Homework 8 solutions

5.3.38 a The general form of a skew-symmetric $3 \times 3$ matrix is $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$, with

$$A^2 = \begin{bmatrix} -a^2 - b^2 & -bc & ac \\ -bc & -a^2 - c^2 & -ab \\ ac & -ab & -b^2 - c^2 \end{bmatrix},$$
a symmetric matrix.

b By Theorem 5.3.9.a, $(A^2)^T = (A^T)^2 = (-A)^2 = A^2$, so that $A^2$ is symmetric.

5.3.42 a Suppose we are projecting onto a subspace $W$ of $\mathbb{R}^n$. Since $A\mathbf{x}$ is in $W$ already, the orthogonal projection of $A\mathbf{x}$ onto $W$ is just $A\mathbf{x}$ itself: $A(A\mathbf{x}) = A\mathbf{x}$, or $A^2\mathbf{x} = A\mathbf{x}$.

Since this equation holds for all $\mathbf{x}$, we have $A^2 = A$.

b $A = QQ^T$, for some matrix $Q$ with orthonormal columns $\mathbf{u}_1, \ldots, \mathbf{u}_m$. Note that $Q^TQ = I_m$, since the $ij$th entry of $Q^TQ$ is $\mathbf{u}_i \cdot \mathbf{u}_j$. Then $A^2 = QQ^TQQ^T = Q(Q^TQ)Q^T = QI_mQ^T = QQ^T = A$.

5.4.4 By Theorem 5.4.1, the equation $(\text{im} \ B)^{\perp} = \ker(B^T)$ holds for any matrix $B$. Now let $B = A^T$. Then $(\text{im} \ A^T)^{\perp} = \ker(A)$. Taking transposes of both sides and using Theorem 5.1.8d we obtain $\text{im}(A^T) = (\ker(A))^\perp$, as claimed.

5.4.6 Yes! For any matrix $A$,

$$\text{im}(A) = (\ker(A^T))^{\perp} = (\ker(AA^T))^{\perp} = (\ker(AA^T)^T)^{\perp} = \text{im}(AA^T).$$

Theorem 5.4.1 Theorem 5.4.2a Theorems 5.4.1 and 5.1.8d.

5.4.22 Using Theorem 5.4.6, we find $\mathbf{x}^* = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{b} - A\mathbf{x}^* = \mathbf{0}$. This system is in fact consistent and $\mathbf{x}^*$ is the exact solution; the error $\|\mathbf{b} - A\mathbf{x}^*\|$ is 0.

5.4.26 Here, the normal equation $A^TA\mathbf{x} = A^T\mathbf{b}$ is

$$\begin{bmatrix} 66 & 78 & 90 \\ 78 & 93 & 108 \\ 90 & 108 & 126 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

with solutions $\mathbf{x}^* = \begin{bmatrix} t - \frac{7}{3} \\ 1 - 2t \\ t \end{bmatrix}$, where $t$ is an arbitrary constant.
5.4.30 We attempt to solve the system
\[
\begin{align*}
0 + 0c_1 &= 0 \\
0 + 0c_1 &= 1, \text{or} \\
0 + 1c_1 &= 1
\end{align*}
\]
\[
\begin{bmatrix}
1 & 0 \\
1 & 0 \\
1 & 1
\end{bmatrix}
\begin{bmatrix}
c_0 \\
c_1
\end{bmatrix}
=
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}.
\]
This system cannot be solved exactly; the least-squares solution is
\[
\begin{bmatrix}
c_0^* \\
c_1^*
\end{bmatrix} = \begin{bmatrix}
\frac{3}{2} \\
\frac{3}{2}
\end{bmatrix}.
\]
The line that fits the data points best is \( f^*(t) = \frac{1}{2} + \frac{1}{2} t \).

Figure 5.18: for Problem 5.4.30.

The line goes through the point (1, 1) and "splits the difference" between (0, 0) and (0, 1). See Figure 5.18.

5.5.8 By parts b and c of Definition 5.5.1, we have \( T(u + v) = \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle = T(u) + T(v) \) and \( T(cv) = \langle cv, w \rangle = c \langle v, w \rangle = c T(v) \), so that \( T \) is linear. If \( w = 0 \), then \( \text{im}(T) = \{0\} \) and \( \ker(T) = V \). If \( w \neq 0 \), then \( \text{im}(T) = \mathbb{R} \) and \( \ker(T) \) consists of all \( v \) perpendicular to \( w \).

5.5.14 a This is not an inner product since there are nonzero polynomials \( f(t) \) in \( P_2 \) with \( f(1) = f(2) = 0 \), so that \( (f, f) = (f(1))^2 + (f(2))^2 = 0 \). (For example, let \( f(t) = (t - 1)(t - 2) \).)

b This is an inner product. We leave it to the reader to check axioms a to c. As for d: A nonzero polynomial \( f \) in \( P_2 \) has at most two zeros, so that \( f(1) \neq 0 \) or \( f(2) \neq 0 \) or \( f(3) \neq 0 \), and \( (f, f) = (f(1))^2 + (f(2))^2 + (f(3))^2 > 0 \).