If we have the equation \( z = 1 + 2x^2 + 2y^2 \), we know this is gonna behave like some type of paraboloid (circular in fact, cause the coefficients for the squared terms are both the same). We further know that since we have that lonely +1 out there, this is just gonna be like a regular circular paraboloid shifted up one unit on the z – axis.

You should get something that looks like this (plotted in Matlab):

These types of problems aren’t that bad if you use process of elimination. I’m going to do them in the order I did them in, not numerically.

**14.1.60: A / 4 (IV)**

I chose to start with this one cause it’s the only one with an exponential in it (so the graph should look significantly different from all the others, esp. the ones with just trig terms in them). \( e^x \) is going to dominate this function over \( \cos y \) for large x (by large I’m talking about \( x > 1 \)), thus giving us an exponential graph with cosine-like oscillations. At small x (anything less than \( x = 1 \)), the function’ll be dominated by \( \cos y \), giving us pretty low values for z.

The only graph that fits this description is A. In terms of level curves, the huge bumpy parts of A are pretty iconic to the curves on number 4 (IV).

**14.1.63: B / 6 (VI)**

Let’s go take care of the non-trig graphs next, namely, 63 and 64. Since trig terms have really repetitive behavior graphically, it’s usually straightforward to weed out the non-trig graphs. Here, the non-trig graphs’ll be B and D.

I looked at 63 first. Note how that along the x-plane \( x = 0 \), the function’ll behave as \( z = 1 - y^2 \) which has a maximum at \( y = 0 \), and along \( y = 0 \), you’ll have a similar thing in the x – direction going on. The only graph that has this is B. That corresponds to level curve set 6 (VI) – look at the hump in the center of the surface, it’s the only one that matches.

You could also do this question by looking at what z – value B or D has at \( (0,0) \). Plugging in zero for x and y tells us that the point \( (0,0,1) \) is on the surface somewhere. Likewise, you could have done this to show that at \( (0,0) \), number 64 will give you the point \( (0,0,0) \), which corresponds to graph D and level curve set 5 (look for the two peaks on the surface).

**14.1.65: C / 2 (II)**

Now let’s go back for the non-trig ones. We only have C, E, and F left. \( z = \sin(xy) \) will always be zero along the x and y axes, so we’re stuck with C for this one. The hyperbolic ridges on this graph kinda give it away as to which level curve set it is: 2 (II)
14.1.61: F / 1 (I)  
Note how this function’ll be zero along the line $y = x$. If we follow the peaks and valleys of this function, we get that 1 is the only one that fits this guy.

14.1.62: E / 3 (III)  
The only combo that’s left. You can tell E matches up with 3 because of all the peaks on the graph, so that’s a start. If you didn’t end with this one though, 62 has to be E cause along the $x = 0$ plane, the function’ll just behave like a regular sinusoidal, and likewise along the $y = 0$ plane.

14.2.10  
Know how to do these problems, these are really common exam questions! Usually, the limit fails to exist because as you approach the point in question from two different directions / lines, you’ll get two different limits. Generally speaking, the usual suspects to try out first are $x = 0$ and $y = 0$. After this, try $y = x$ or its relative, $y = mx$, where $m$ is any real number your heart desires.

This last one, $y = mx$, is a pretty useful one because as long as you have an $m$ left over in your limit after you’ve simplified things, you’ve shown that the thing that the limit approaches is dependent on a number that you choose. Hint: this works wonders on tests. And on this problem. It’s just cool like that.

Also before I forget, you should always write $\lim_{(x,y) \to (0,0)}$ or whatever when you’re doing these problems for each step cause it’s mathematically correct. Yeah, it’s a pain, but hey, I didn’t make the rules. Anyways, if we use $y = mx$, we get that:

\[
y = mx: \lim_{(x,y) \to (0,0)} \frac{5y^4 \cos^2 x}{x^4 + y^4} = \lim_{(x,y) \to (0,0)} \frac{5m^4 x^4 \cos^2 x}{x^4(1 + m^4)} = \lim_{(x,y) \to (0,0)} \frac{5m^4 \cos^2 x}{(1 + m^4)} = \frac{5m^4}{(1 + m^4)}
\]

Don’t forget that $\cos(0) = 1$. **Since the limit depends on $m$, it doesn’t exist.**

14.2.13  
Let’s start with $y = x$ and $y = x^2$ since $y = 0$ and $x = 0$ aren’t very useful here:

\[
y = x: \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{x^2}{\sqrt{2x^2}} = 0
\]

\[
y = x^2: \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{x^3}{\sqrt{x^2 + x^4}} = \lim_{(x,y) \to (0,0)} \frac{x^2}{\sqrt{1 + x^2}} = 0
\]

It looks like everything’s equaling zero. In fact, since the power of the bottom is always going to be larger than the top, and the top is just composed of $xy$, you’re never gonna have a case where you can have the top cancelling out and get something other than an answer that doesn’t exist or zero.

If you ever get to the point where you’re trying larger and larger exponents and things just ain’t working out, try a coordinate change.

\[
\lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} = \lim_{(r,\theta) \to (0,0)} \frac{r^2 \cos \theta \sin \theta}{r} = \lim_{(r,\theta) \to (0,0)} r \cos \theta \sin \theta = 0
\]

That last limit on the right isn’t able to be simplified any further so **the limit exists and is zero.**
14.2.14

This one exists! Note that you can do factoring on the top because \((x^4 - y^4) = (x^2 + y^2)(x^2 - y^2)\) and the first term on the right there cancels with the bottom. So, we’re left with the following (really straightforward) limit:

\[
\lim_{(x,y) \to (0,0)} (x^2 - y^2) = 0
\]

14.2.15

If you notice, we can get everything in terms of \(x^4\) if we set \(y = x^2\). We can do this with our usual suspect \(y = x\).

\[
\begin{align*}
y &= x: \quad \lim_{(x,y) \to (0,0)} \frac{x^2 y e^y}{x^4 + 4y^2} &= \lim_{(x,y) \to (0,0)} \frac{x^3 e^x}{x^4 + 4x^2} = \lim_{(x,y) \to (0,0)} \frac{x e^x}{x^2 + 4} = 0 \\
y &= x^2: \quad \lim_{(x,y) \to (0,0)} \frac{x^2 y e^y}{x^4 + 4y^2} &= \lim_{(x,y) \to (0,0)} \frac{x^4 e^{x^2}}{5x^4} = \frac{1}{5}
\end{align*}
\]

Since zero can’t equal one fifth, the limit doesn’t exist.

14.2.18

For this one we can do \(y = x\) and \(x = y^4\).

\[
\begin{align*}
y &= x: \quad \lim_{(x,y) \to (0,0)} \frac{xy^4}{x^2 + y^8} &= \lim_{(x,y) \to (0,0)} \frac{x^5}{x^2 + x^8} = \lim_{(x,y) \to (0,0)} \frac{x^3}{1 + x^6} = 0 \\
x &= y^4: \quad \lim_{(x,y) \to (0,0)} \frac{xy^4}{x^2 + y^8} &= \lim_{(x,y) \to (0,0)} \frac{y^8}{2y^8} = \frac{1}{2}
\end{align*}
\]

So the limit doesn’t exist because zero doesn’t equal one half.

14.2.32

We just can’t have the bottom be zero, which’ll happen wherever \(e^{xy} = 1\). This in turn happens wherever \(x = 0\) or \(y = 0\). So our domain is just \((x,y) \in \mathbb{R} \ s.t. \ x, y \neq 0\).

14.2.37

If we have a piecewise function that’s made up of continuous bits, we just need to look at the places where the functions are joined for discontinuities. Here, \(\frac{x^3 y^3}{2x^2 + y^2}\) is continuous everywhere except at \((x,y) = (0,0)\). If the limit as \((x,y)\) goes to \((0,0)\) indeed approaches 1 though, we’re okay and \(f(x)\)'ll be continuous everywhere. Otherwise, we’ll have a discontinuity at the origin. This question is really asking if \(\lim_{(x,y) \to (0,0)} \frac{x^3 y^3}{2x^2 + y^2}\) equals 1 or not. So, let’s go with the usual suspect: \(y = x\).

\[
\begin{align*}
y &= x: \quad \lim_{(x,y) \to (0,0)} \frac{x^2 y^3}{2x^2 + y^2} &= \lim_{(x,y) \to (0,0)} \frac{x^5}{3x^2} = 0
\end{align*}
\]

Actually, we can stop here (aka, should, cause we’re lazy) because even if the limit exists, it'll have to equal 0, not 1. So, our function’ll be continuous everywhere except at the origin: \((x,y) \in \mathbb{R} \ s.t. \ (x,y) \neq (0,0)\).
Not gonna lie, this section’s long, but plain old partial derivatives aren’t that bad so long as you’re still up to speed with your derivative rules. It’s never too late to review!

\[
\begin{align*}
\frac{\partial w}{\partial u} &= -\frac{e^v}{(u + v^2)^2} \\
\frac{\partial w}{\partial v} &= \frac{(u + v^2)e^v - 2ve^v}{(u + v^2)^2}
\end{align*}
\]

\[\frac{\partial u}{\partial r} = (\cos \theta)(\cos(r \cos \theta)) \quad \frac{\partial u}{\partial \theta} = r(\cos(r \cos \theta))\]

\[
\frac{\partial w}{\partial x} = yz^2e^{xyz} \quad \frac{\partial w}{\partial y} = xz^2e^{xyz} \quad \frac{\partial w}{\partial z} = xyz e^{xyz} + e^{xyz} \rightarrow \frac{\partial w}{\partial z} = e^{xyz}(1 + xyz)
\]

We just have to plug numbers in now. Le gasp. Just don’t forget about the chain rule down there!

\[
f_z = \frac{2 \sin z \cos z}{2\sqrt{\sin^2 x + \sin^2 y + \sin^2 z}} \rightarrow f_z(0,0,\frac{\pi}{4}) = \frac{\sin(\frac{\pi}{4})\cos(\frac{\pi}{4})}{\sin^2(\frac{\pi}{4})} = \cot(\frac{\pi}{4}) \rightarrow f_z(0,0,\frac{\pi}{4}) = 1
\]

Implicit differentiation with three variables works really similarly to when we used to do it with two variables. Just now, if we’re doing things like, \(\frac{\partial z}{\partial x}\), we get to keep \(y\) constant.

\[
y \frac{\partial z}{\partial x} + \ln y = 2z \frac{\partial z}{\partial x} \rightarrow \frac{\partial z}{\partial x} = \frac{\ln y}{2z - y}
\]

\[
y \frac{\partial z}{\partial y} + z + \frac{x}{y} = 2z \frac{\partial z}{\partial y} \rightarrow \frac{\partial z}{\partial y} = \frac{z + \frac{x}{y}}{2z - y}
\]

A: The basic idea is to not forget what you’re differentiating with respect to.

\[
\frac{\partial z}{\partial x} = g(y)f'(x) \quad \frac{\partial z}{\partial y} = f(x)g'(y)
\]

B: Don’t forget the chain rule here.

\[
\frac{\partial z}{\partial x} = yf'(xy) \quad \frac{\partial z}{\partial y} = xf'(xy)
\]

C: Again, the chain rule’s your best friend. At least for the time being.

\[
\frac{\partial z}{\partial x} = \frac{f'(x/y)}{y} \quad \frac{\partial z}{\partial y} = -\frac{x}{y^2}f'(x/y)
\]
Aren’t you tired of taking derivatives yet? I sure am.

\[ f_x = 2m \sin(mx + ny) \cos(mx + ny) \quad f_y = 2n \sin(mx + ny) \cos(mx + ny) \]
\[ f_{xx} = 2m^2 \cos(mx + ny) \cos(mx + ny) - 2m^2 \sin(mx + ny) \sin(mx + ny) \]
\[ f_{xx} = 2m^2 (\cos^2(mx + ny) - \sin^2(mx + ny)) \rightarrow f_{xx} = 2m^2 \cos(2mx + 2ny) \]

The case with the y's is the same, just with an n out front instead of an m. Using the same steps, you get:

\[ f_{yy} = 2n^2 \cos(2mx + 2ny) \]

Because of Fubini’s theorem, the mixed partials are equal to each other. You get the same thing in essence:

\[ f_{xy} = f_{yx} = 2mn \cos(2mx + 2ny) \]

Lol, guess we have to prove what we used for number 54 down here.

\[ u_x = 4x^3 y^3 \quad u_y = 3x^4 y^2 - 4y^3 \]
\[ 4u_{xy} = 12x^3 y^2 \quad u_{yx} = 12x^3 y^2 \rightarrow u_{xy} = u_{yx} \]

\[ g_r = e^r \sin(st) \rightarrow g_{rs} = te^r \cos(st) \rightarrow g_{rst} = e^r \cos(st) - st \sin(st) \]

If we differentiate with respect to y first, we don’t have to deal with that pain in the butt inverse sine term at all.

\[ f_y = 2xyz^3 \rightarrow f_{yx} = 2yz^3 \rightarrow f_{xy} = f_{xx} = 6yz^2 \]

Last problem! This one’s pretty straightforward.

\[ u_t = -a^2 k^2 e^{-a^2 k^2 t} \sin(kx) \]
\[ u_x = ke^{-a^2 k^2 t} \cos(kx) \rightarrow u_{xx} = -k^2 e^{-a^2 k^2 t} \sin(kx) \]

So:

\[ u_t = a^2 u_{xx} \]

Congratulations if you did this all this differentiation without going crazy.