You should really memorize the formula for the length of a curve in space. If you forget it, just think of it as Pythagorean Theorem in 3D and summing the little hypotenuses together.

\[ L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt \]

Anyways, now to the problem. Usually there’s gonna be some sort of trick of simplifying the stuff under the square root so you can actually integrate the thing. The first thing to check for is whether the radicand is the square of something else. If that doesn’t work, completing the square sometimes works.

\[ L = \int_{0}^{1} \sqrt{4 + 4t^2 + t^4} \, dt = \int_{0}^{1} \sqrt{(t^2 + 2)^2} \, dt = \frac{1}{3} t^3 + 2t \bigg|_{0}^{1} \rightarrow L = \frac{7}{3} \]

Don’t forget the chain rule for the k term and your trig rules! You can do a u substitution on the

\[ L = \int_{0}^{\pi} \sqrt{\sin^2 t + \cos^2 t + \left(\frac{-\sin t}{\cos t}\right)^2} \, dt = \int_{0}^{\pi} \sqrt{1 + \tan^2 t} \, dt = \int_{0}^{\pi} \sec t \, dt \]

Integrating secant is an interesting trick that you might’ve learned way back in calc II. The trick is to multiply the top and bottom by the quantity \( \sec t + \tan t \) and then do a u-substitution on that quantity.

\[ L = \int_{0}^{\pi} \frac{\sec t \left(\sec t + \tan t\right)}{\sec t + \tan t} \, dt \rightarrow u = \sec t + \tan t, \, du = (\sec t \tan t + \sec^2 t) \, dt \]

Notice that if we multiply out the top, we just get \( du \)! Do it out if this isn’t super clear. Anyways, we can integrate to get:

\[ L = \ln(\sec t + \tan t) \bigg|_{0}^{\pi} = \ln(\sqrt{2} + 1) - \ln(1 + 0) \rightarrow L = \ln(\sqrt{2} + 1) \]

This’d make a pretty good exam question by the way.

This problem works really similarly to the one in the book on page 855. Reparameterizing a curve with respect to arc length sounds like a bunch of extra work, but reparameterizing things in general sometimes can make things easier to solve. I mean, we’re gonna have to integrate an square root for these types of equations right? If we can make the stuff under the radical simpler, that’d be cool.

The methodology behind this question’s like, find \( ds/dt \), where \( ds \) is a tiny bit of the arc of the curve. We can then integrate to find the relationship between \( s \) and \( t \), and then solve for \( t \) so we get a function of \( t \) in terms of \( s \). Finally, we plug in our function for \( t \) in terms of \( s \) and we’re done.
If you ever have a final result where you don’t have all s’s in there instead of t’s, something’s fishy. Anyways, let’s get going on the problem. Note that for $ds/dt$, we’re not doing any kind of calculus yet: we’re just getting the infinitesimal length of one of the arc bits.

$$\frac{ds}{dt} = \sqrt{e^{4t} \cos^2 2t + 4 + e^{4t} \sin^2 2t} = e^{2t} \sqrt{\cos^2 2t + \sin^2 2t + 4} = e^{2t} \sqrt{5}$$

For the integration, make sure to integrate over a dummy variable (let’s choose $u$ like the book did), and replace all the t’s with that dummy variable.

$$s = \int_0^t \frac{ds}{dt} \, du = \int_0^t e^{2u} \sqrt{5} \, du = \frac{1}{2} e^{2t} \sqrt{5} + 1$$

$$t = \frac{1}{2} \ln \frac{2s - 2}{\sqrt{5}}$$

Don’t forget the plus one out there! Now we need to plug this back into $\vec{r}(t) = (e^{2t} \cos 2t , 2e^{2t} \sin 2t)$

$$\vec{r}(s) = \left\langle \frac{2s - 2}{\sqrt{5}} \cos \left( \ln \frac{2s - 2}{\sqrt{5}} \right), 2, \frac{2s - 2}{\sqrt{5}} \sin \left( \ln \frac{2s - 2}{\sqrt{5}} \right) \right\rangle$$

13.3.18

These questions tend to take a while, so make sure you know the steps so you can do ’em quickly! I mean, just in case they’re coming to a test near you.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle 2t, \cos t + t \sin t - \cos t, -\sin t + t \cos t + \sin t \rangle}{\sqrt{4t^2 + t^2 \sin^2 t + t^2 \cos^2 t}} = \frac{\langle 2t, t \sin t, t \cos t \rangle}{\sqrt{5}t^2}$$

$$\vec{T}(t) = \left\langle \frac{2t}{t \sqrt{5}} \sin t, t \cos t \right\rangle \rightarrow \vec{T}(t) = \frac{1}{\sqrt{5}} \langle 2, \sin t, \cos t \rangle$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{1}{\sqrt{5}} \langle 0, \cos t, -\sin t \rangle \rightarrow \vec{N}(t) = \langle 0, \cos t, -\sin t \rangle$$

And last but not least, we have to get the curvature.

$$\kappa(t) = \frac{|\vec{T}'(t) \times \vec{T}''(t)|}{|\vec{T}'(t)|^3} = \frac{|\vec{T}'(t)|}{|\vec{T}'(t)|} \cdot \frac{1}{\sqrt{5} t^{\frac{3}{2}}} \rightarrow \kappa(t) = \frac{1}{5t}$$

13.3.19

Not gonna lie, this problem’s tricky because it can be horrendously miserable unless you spot the fact that $2 + e^{2t} + e^{-2t}$ can be rewritten as $(e^t + e^{-t})^2$ (stare at it for a few seconds, it works out). I also factored out an $e^t$ from the bottom of the y component and cancelled it out with the top because it makes things a bit easier when you’re getting T’(t). Likewise, you can take out an $e^{-t}$ for the z component.

$$\vec{T}(t) = \frac{\langle \sqrt{2}, e^t, -e^{-t} \rangle}{\sqrt{2} + e^{2t} + e^{-2t}} = \frac{\langle \sqrt{2}, e^t, -e^{-t} \rangle}{e^t + e^{-t}} = \langle \frac{\sqrt{2}}{e^t e^{-t}}, -\frac{1}{e^{2t}} - \frac{1}{e^{2t}}, 1 \rangle \rightarrow \vec{T}(t) = \langle \frac{\sqrt{2}}{e^t e^{-t}}, e^{2t}, -e^{-2t} \rangle$$
Getting the unit normal becomes hella nasty, but remember that unless you’re specifically asked to, you don’t have to simplify this thing.

\[ \vec{N}(t) = \frac{-\sqrt{2}e^t + \sqrt{2}e^{-t}}{(e^t + e^{-t})^2}, 2e^{2t}, 2e^{-2t} \]

\[ \frac{2e^{2t} - 4 + 2e^{-2t}}{(e^t + e^{-t})^4} + 4e^{4t} + 4e^{-4t} \]

\[ \kappa(t) = \frac{\sqrt{2e^{2t} - 4 + 2e^{-2t}} + 4e^{4t} + 4e^{-4t}}{2 + e^{2t} + e^{-2t}} \to \kappa(t) = \frac{\sqrt{2e^{2t} - 4 + 2e^{-2t}} + 4e^{4t} + 4e^{-4t}}{e^t + e^{-t}} \]

13.3.30

This is a function with just one variable so we can use the form of \( \tilde{\kappa}(t) \) on page 857:

\[ \tilde{\kappa}(t) = \frac{|y''|}{(1 + (y')^2)^{3/2}} \]

\[ y' = 1/x \to y'' = -1/x^2 \to \tilde{\kappa}(t) = \frac{1/x^2}{(1 + \frac{1}{x^2})^{3/2}} = \frac{1}{x^2(1 + x^{-2})^{3/2}} \]

Now we can do the first derivative test on this guy:

\[ \kappa'(t) = \frac{-2x(1 + x^{-2})^{3/2} + \frac{3}{2}x^2(1 + x^{-2})^2(-2x^{-3})}{x^4(1 + x^{-2})^3} = \frac{-2(1 + x^{-2}) + \frac{3}{2}x(-2x^{-3})}{x^3(1 + x^{-2})^{5/2}} \]

\[ \kappa'(t) = \frac{-2 + x^{-2}}{x^3(1 + x^{-2})^{5/2}} = 0 \to x = \pm \frac{1}{\sqrt{2}} \]

But, we know that natural logarithms only exist for positive x, so we get one extrema at \( x = \sqrt{2} \). We know this is a maximum because \( \tilde{\kappa}'(t) \) is positive for \( x < \sqrt{2} \) and positive otherwise. So to get our point, we just have to plug this back into y to get:

\[ (x, y) = \left( \frac{1}{\sqrt{2}}, \ln \frac{1}{\sqrt{2}} \right) \]

13.3.33

The best way to do this question I think would just be to trace the plot onto your paper and do all the sketching there. This isn’t really a math question honestly. Anyways, the curvature’s greater at P because the radius of the osculating circle’s smaller.

I’ve drawn the circles on that picture to the left. We’re only looking for estimations here, but:

\[ r_p \approx 0.6 \to \kappa_p \approx 1.667 \]

\[ r_Q \approx 0.9 \to \kappa_Q \approx 1.111 \]
13.3.48

The point \((1, \frac{2}{3}, 1)\) happens when \(t = 1\).

\[
\overline{T}(t) = \frac{(2t, 2t^2, 1)}{\sqrt{4t^2 + 4t^4 + 1}} = \frac{(2t, 2t^2, 1)}{2t^2 + 1} \rightarrow \overline{T}(1) = \frac{(2,2,1)}{3} \rightarrow \overline{T}(1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)
\]

\[
\overline{T}'(t) = \left(\frac{4t^2 + 2}{(2t^2 + 1)^2}, \frac{4t}{(2t^2 + 1)^2}, \frac{-4t}{(2t^2 + 1)^2}\right) \rightarrow \overline{T}'(1) = \left(-\frac{2}{9}, \frac{4}{9}, -\frac{4}{9}\right)
\]

\[
|\overline{T}'(1)| = \frac{1}{9} \sqrt{4 + 16 + 16} = \frac{2}{3} \rightarrow \overline{N}(1) = \frac{3}{2} \left(-\frac{2}{9}, \frac{4}{9}, -\frac{4}{9}\right) \rightarrow \overline{N}(1) = \left(-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)
\]

13.4.4

This problem's pretty straightforward. I didn't draw the vectors here cause I'm lazy.

\[
\overline{v}(t) = (-1, \frac{2}{\sqrt{t}}) \rightarrow \overline{v}(1) = (-1, 2) \quad \overline{s}(t) = \sqrt{1 + \frac{4}{t}} \quad \overline{a}(t) = (0, -t^{-3}) \rightarrow \overline{a}(1) = (0, -1)
\]

To draw the path, we can do the following. I didn't actually plot the stuff, but you can just put it into a calc.

\[
x = 2 - t \rightarrow t = 2 - x
\]

\[
y = 4\sqrt{t} = 4\sqrt{2 - x} \rightarrow y = \sqrt{32 - 16x}
\]

13.4.8

\[
\overline{v}(t) = (1, -2 \sin t, \cos t) \rightarrow \overline{v}(0) = (1, 0, 1) \quad \overline{a}(t) = (0, -2 \cos t, -\sin t) \rightarrow \overline{a}(0) = (0, -2, 0)
\]

\[
\overline{s}(t) = \sqrt{1 + 4 \sin^2 t + \cos^2 t} \rightarrow \overline{s}(t) = \sqrt{2 + 3 \sin^2 t}
\]

We can do the same thing to draw the path, except this time we can put everything in terms of \(z\) instead.

13.4.10

\[
\overline{v}(t) = (-2 \sin t, 3, 2 \cos t) \quad \overline{a}(t) = (-2 \cos t, 0, -2 \sin t)
\]

\[
\overline{s}(t) = \sqrt{4 \sin^2 t + 3 \cos^2 t + 9} \rightarrow \overline{s}(t) = \sqrt{13}
\]

13.4.16

Now we just go backwards and do some integration. I'm not going to use bracket notation here for vectors though. Since we're going to be adding \(c\) each time we integrate, we should do engineering notation.

\[
\overline{v}(t) = 2t\hat{i} + 3t^2\hat{j} + 4t^3\hat{k} + c
\]

Plugging in our initial conditions, we get that \(c = \hat{i}\). Therefore, \(\overline{v}(t) = (2t + 1)\hat{i} + 3t^2\hat{j} + 4t^3\hat{k}\). Next we integrate again:

\[
\overline{r}(t) = (t^2 + t)\hat{i} + t^3\hat{j} + t^4\hat{k} + c
\]
Our initial conditions point to \( c = \hat{j} - \hat{k}, \) so we can combine those with the second and third terms to get:

\[
\vec{r}(t) = (t^2 + t)\hat{i} + (t^3 + 1)\hat{j} + (t^4 - 1)\hat{k}
\]

13.4.19

Moar first derivative testing it is then.

\[
\vec{v}(t) = (2t, 5, 2t - 16) \rightarrow \vec{s}(t) = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281}
\]

We can't factor this down unfortunately. Anyways, it's all first derivatives from here.

\[
\vec{s}'(t) = \frac{8t - 32}{\sqrt{8t^2 - 64t + 281}} = 0
\]

We don't need to worry about the bottom since we're doing the first derivative test here. We get that the speed reaches some sort of max / min value at \( t = 4. \) We can tell this is a minimum speed though cause the function is negative for \( t < 4 \) and positive for \( t > 4. \) So our answer’s \( t = 4. \)

13.4.22

This one was kinda tricky so if you didn’t get this question, don’t worry about it :) There was a hint for the people using Webassign, and that hint kinda gave it away. For the people on Webassign, they were told to look at \( \vec{v} \cdot \vec{v}. \) Although it’s kinda confusing why you’re looking at a dot product at first, you gotta notice that by definition:

\[
\vec{s}'(t) = \sqrt{\vec{v} \cdot \vec{v}} = v
\]

Notice how on the right hand side is a constant \( v, \) not a vector \( v \) (they said speed was constant right?). This represents the magnitude of velocity We can square both sides to give:

\[
\vec{v} \cdot \vec{v} = v^2
\]

We need to look at the relationship between velocity and acceleration, so we can get to acceleration by just differentiating both sides:

\[
2\vec{v} \cdot \frac{d}{dt} (\vec{v}) = 2v \frac{dv}{dt}
\]

The right side is just zero because \( v \) is a scalar number (magnitude of velocity remember?) and differentiating a scalar’l get you a big fat zero.

\[
2\vec{v} \cdot \frac{d}{dt} (\vec{v}) = 0
\]

Now we can just wrap up since the derivative of velocity is just acceleration:

\[
2\vec{v} \cdot \vec{a} = 0
\]

Since the dot product is zero, the vectors must be orthogonal to each other. Done.
13-4.32

If you’re taking physics right now or have taken physics sometime in the past, this question’s pretty much a breeze. Actually, you don’t need any calculus to do this question the physics way, which is significantly easier than the way your book would like you to do. There are just three equations you need to know for this part:

\[ v_f = v_i + at \]
\[ v_f^2 = v_i^2 + 2ad \]
\[ d = v_i t + \frac{1}{2} at^2 \]

Where \( v_f \) is the final velocity in any one direction, \( v_i \) is the initial velocity in any one direction, \( a \) is the acceleration in that direction, and \( t \) is the time spent moving. We have three directions of movement here (x, y, and z) so we can look at things one at a time in each dimension. Y’know, break things down into its x, y, and z components. Here, we’re gonna let z be east (aka stuff because of the wind).

The first thing we need to look for is the distance in the x direction (south) the ball travels. We’re given the initial velocity of the ball which we can break down into its x and y components:

\[ v_{i,x} = 30 \cos \theta \]
\[ v_{i,y} = 30 \sin \theta \]

We have a velocity in the x direction and know that there's no air resistance in that direction so \( a_x = 0 \). Velocity is just distance over time, so we need a time. To figure out the time, we can look at the behavior of the ball in the y-direction. Since we have projectile motion, the ball’s gonna travel in a parabola. If we take the third equation up there, we see that the distance above ground as a function of time is:

\[ d_y = v_{i,y} t - \frac{1}{2} (9.81 \frac{m}{s^2}) t^2 \]

If we set this y-distance to zero, we can see that the ball is at zero height at \( t = 0 \) (when we threw it) and at \( t = 3.058 \) s. So, the ball spends a grand total of 3.058 seconds in the air. We can now use this to find horizontal distance. Once again, notice that there’s no acceleration in the x direction so the second term goes away.

\[ d_x = v_{i,x} t + \frac{1}{2} a_x t^2 \rightarrow d_x = (3.058 \text{ s})(30 \cos \frac{\pi}{6}) = 79.449 \text{ m} \]

Now for the z direction. The initial velocity in the z direction is zero and but there’s acceleration in that direction. We know that \( F = ma \) so \( a = F/m \). We get that:

\[ d_z = \frac{1}{2} \left( \frac{4 \text{ N}}{0.8 \text{ kg}} \right) (3.058 \text{ s})^2 = 23.378 \text{ m} \]

The total distance the ball travelled is therefore:

\[ d_{total} = \sqrt{d_x^2 + d_z^2} = \sqrt{(79.449 \text{ m})^2 + (23.378 \text{ m})^2} \rightarrow d_{total} = 82.816 \text{ m} \]

The final velocity can be calculated by \( v_{total} = \sqrt{v_{f,x}^2 + v_{f,y}^2 + v_{f,z}^2} \). We already established that there’s no change in velocity for the x-component, so \( v_{f,x} = v_{i,x} \). Also if we have parabolic motion and the starting
height of the ball equals the ending height of the ball, \( v_{f,y} = -v_{i,y} \). The only thing we really need to find is the \( z \)-component of the velocity. This is easiest done with the first equation:

\[
v_{f,z} = v_{i,z} + at = \left( \frac{4 \text{ N}}{0.8 \text{ kg}} \right) (3.058 \text{ s}) = 15.29 \frac{\text{m}}{\text{s}}
\]

\[
v_{\text{total}} = \sqrt{(25.981 \text{ m/s})^2 + (-15 \text{ m/s})^2 + (15.29 \text{ m/s})^2} \rightarrow v_{\text{total}} = 33.672 \frac{\text{m}}{\text{s}}
\]

That’s a ridiculously fast velocity (that’s about 75 mph in English units) btw. But then, this is math class.

13.4.36

More proofs! The first one’s kinda intuitive: if you’re going in a straight line, your **acceleration vector must be parallel to the velocity vector at all times**. The second one’s not that bad if you got number 22 back there or if you’ve taken physics: **your acceleration vector’ll be perpendicular to your velocity vector** as it’s going through that curve.