6. \( \mathbf{a} \times \mathbf{b} = \begin{vmatrix} i & j & k \\ t & \cos t & \sin t \\ 1 & -\sin t & \cos t \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} i - \begin{vmatrix} t & \sin t \\ 1 & \cos t \end{vmatrix} j + \begin{vmatrix} t & \cos t \\ 1 & -\sin t \end{vmatrix} k \)

\[= [\cos^2 t - (\sin^2 t)] i - (t \cos t - \sin t) j + (-t \sin t - \cos t) k = i + (\sin t - t \cos t) j + (-t \sin t - \cos t) k \]

Since

\[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = [i + (\sin t - t \cos t) j + (-t \sin t - \cos t) k] \cdot (t i + \cos t j + \sin t k) \]

\[= t + \sin t \cos t - t \cos^2 t - t \sin^2 t - \sin t \cos t \]

\[= t - t (\cos^2 t + \sin^2 t) = 0 \]

\( \mathbf{a} \times \mathbf{b} \) is orthogonal to \( \mathbf{a} \).

Since

\[(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = [i + (\sin t - t \cos t) j + (-t \sin t - \cos t) k] \cdot (i - \sin t j + \cos t k) \]

\[= 1 - \sin^2 t + t \sin t \cos t - t \sin t \cos t - \cos^2 t \]

\[= 1 - (\sin^2 t + \cos^2 t) = 0 \]

\( \mathbf{a} \times \mathbf{b} \) is orthogonal to \( \mathbf{b} \).

14. Using Theorem 9, we have \(|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta = (4)(5) \sin 45^\circ = 20 \cdot \frac{\sqrt{2}}{2} = 10 \sqrt{2} \). By the right-hand rule, \( \mathbf{u} \times \mathbf{v} \) is directed out of the page.

32. (a) \( \overrightarrow{PQ} = \langle 1, 2, 1 \rangle \) and \( \overrightarrow{PR} = \langle 5, 0, -2 \rangle \), so a vector orthogonal to the plane through \( P, Q, \) and \( R \) is

\[\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2(-2) - (1)(0), (1)(5) - (1)(-2), (1)(0) - (2)(5) \rangle = \langle -4, 7, -10 \rangle \text{ [or any scalar multiple thereof]} \]

(b) The area of the parallelogram determined by \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \) is \( |\overrightarrow{PQ} \times \overrightarrow{PR}| = |-4, 7, -10| = \sqrt{16 + 49 + 100} = \sqrt{165} \), so the area of triangle \( PQR \) is \( \frac{1}{2} \sqrt{165} \).

36. \( \mathbf{a} = \overrightarrow{PQ} = \langle -4, 2, 4 \rangle \), \( \mathbf{b} = \overrightarrow{PR} = \langle 2, 1, -2 \rangle \) and \( \mathbf{c} = \overrightarrow{PS} = \langle -3, 4, 1 \rangle \).

\[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = -4 \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} = -36 + 8 + 44 = 16 \), so the volume of the parallelepiped is 16 cubic units.
40. \[ |r| = \sqrt{4^2 + 4^2} = 4\sqrt{2} \text{ ft}. \] A line drawn from the point \( P \) to the point of application of the force makes an angle of \( 180^\circ - (45 + 30)^\circ = 105^\circ \) with the force vector. Therefore, \[ |\tau| = |r \times F| = |r| |F| \sin \theta = (4 \sqrt{2}) \times (36) \sin 105^\circ \approx 197 \text{ ft-lb}. \]

45. (a) The distance between a point and a plane is the length of the perpendicular from the point to the plane, here \( |\overrightarrow{T P}| = d \). But \( \overrightarrow{T P} \) is parallel to \( b \times a \) (because \( b \times a \) is perpendicular to \( b \) and \( a \)) and \( d = |\overrightarrow{T P}| = \) the absolute value of the scalar projection of \( c \) along \( b \times a \), which is \( |c| \cos \theta \). (Notice that this is the same setup as the development of the volume of a parallelepiped with \( h = |c| \cos \theta \)). Thus \( d = |c| \cos \theta = h = V/A \) where \( A = |a \times b|, \) the area of the base. So finally \( d = \frac{V}{A} = \frac{|a \cdot (b \times c)|}{|a \times b|} \).

(b) \( n = \overrightarrow{QR} = \langle -1, 2, 0 \rangle, \) \( b = \overrightarrow{QS} = \langle -1, 0, 3 \rangle \) and \( c = \overrightarrow{QP} = \langle 1, 1, 4 \rangle \). Then

\[ a \cdot (b \times c) = \begin{vmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \\ 1 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} + 0 = 17 \]

and \[ a \times b = \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} i + \begin{vmatrix} -1 & 3 \\ -1 & 0 \end{vmatrix} j + \begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix} k = 6i + 3j + 2k \]

Thus \( d = \frac{|a \cdot (b \times c)|}{|a \times b|} = \frac{17}{\sqrt{36 + 9 + 4}} = \frac{17}{7} \).

4. This line has the same direction as the given line, \( v = 2i - 3j + 9k \). Here \( r_0 = 14j - 10k \), so a vector equation is \( r = (14 \mathbf{j} - 10 \mathbf{k}) + t(2 \mathbf{i} - 3 \mathbf{j} + 9 \mathbf{k}) = 2t \mathbf{i} + (14 - 3t) \mathbf{j} + (-10 + 9t) \mathbf{k} \) and parametric equations are \( x = 2t, \)
\( y = 14 - 3t, z = -10 + 9t \).

12. Setting \( z = 0 \) we see that \( (1, 0, 0) \) satisfies the equations of both planes, so they do in fact have a line of intersection.

The line is perpendicular to the normal vectors of both planes, so a direction vector for the line is \( \mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 2, 3 \rangle \times \langle 1, -1, 1 \rangle = \langle 5, 2, -3 \rangle \). Taking the point \( (1, 0, 0) \) as \( P_0 \), parametric equations are \( x = 1 + 5t, \)
\( y = 2t, z = -3t, \) and symmetric equations are \( \frac{x - 1}{5} = \frac{y}{2} = \frac{z}{-3}. \)
16. (a) A vector normal to the plane \( x - y + 3z = 7 \) is \( \mathbf{n} = (1, -1, 3) \), and since the line is to be perpendicular to the plane, \( \mathbf{n} \) is also a direction vector for the line. Thus parametric equations of the line are \( x = 2 + t, \ y = 4 - t, \ z = 6 + 3t \).

(b) On the \( xy \)-plane, \( z = 0 \). So \( z = 6 + 3t = 0 \Rightarrow t = -2 \) in the parametric equations of the line, and therefore \( x = 0 \) and \( y = 6 \), giving the point of intersection \((0, 6, 0)\). For the \( yz \)-plane, \( x = 0 \) so we get the same point of intersection \((0, 6, 0)\). For the \( xz \)-plane, \( y = 0 \) which implies \( t = 4 \), so \( x = 6 \) and \( z = 18 \) and the point of intersection is \((6, 0, 18)\).

20. Since the direction vectors are \( \mathbf{v}_1 = (-12, 9, -3) \) and \( \mathbf{v}_2 = (8, -6, 2) \), we have \( \mathbf{v}_1 = -\frac{3}{2} \mathbf{v}_2 \) so the lines are parallel.

28. Since the two planes are parallel, they will have the same normal vectors. A normal vector for the plane \( z = x + y \) or \( x + y - z = 0 \) is \( \mathbf{n} = (1, 1, -1) \), and an equation of the desired plane is \( 1(x - 2) + 1(y - 4) - 1(z - 6) = 0 \) or \( x + y - z = 0 \) (the same plane!).

38. The points \((0, -2, 5)\) and \((-1, 3, 1)\) lie in the desired plane, so the vector \( \mathbf{v}_1 = (-1, 5, -4) \) connecting them is parallel to the plane. The desired plane is perpendicular to the plane \( 2x = 5x + 4y \) or \( 5x + 4y - 2z = 0 \) and for perpendicular planes, a normal vector for one plane is parallel to the other plane, so \( \mathbf{v}_2 = (5, 4, -2) \) is also parallel to the desired plane.

A normal vector to the desired plane is \( \mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (-10 + 16, -20 - 2, -4 - 25) = (6, -22, -29) \). Taking \((x_0, y_0, z_0) = (0, -2, 5)\), the equation we are looking for is \( 6(x - 0) - 22(y + 2) - 29(z - 5) = 0 \) or \( 6x - 22y - 29z = -101 \).

56. The normal vectors are \( \mathbf{n}_1 = (1, 2, 2) \) and \( \mathbf{n}_2 = (2, -1, 2) \). The normals are not parallel, so neither are the planes.

Furthermore, \( \mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 2 + 4 = 4 \neq 0 \), so the planes aren’t perpendicular. The angle between them is given by

\[
\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|} = \frac{4}{\sqrt{9} \sqrt{9}} = \frac{4}{9} \Rightarrow \theta = \cos^{-1} \left( \frac{4}{9} \right) \approx 63.6^\circ.
\]

64. (a) For the lines to intersect, we must be able to find one value of \( t \) and one value of \( s \) satisfying the three equations

\[1 + t = 2 - s, \quad 1 - t = s \quad \text{and} \quad 2t = 2.\]

From the third we get \( t = 1 \), and putting this in the second gives \( s = 0 \). These values of \( s \) and \( t \) do satisfy the first equation, so the lines intersect at the point \( P_0 = (1 + 1, 1 - 1, 2(1)) = (2, 0, 2) \).

(b) The direction vectors of the lines are \( (1, -1, 2) \) and \( (-1, 1, 0) \), so a normal vector for the plane is

\[ (-1, 1, 0) \times (1, -1, 2) = (2, 2, 0) \]

and it contains the point \((2, 0, 2)\). Then an equation of the plane is

\[ 2(x - 2) + 2(y - 0) + 0(z - 2) = 0 \quad \Rightarrow \quad x + y = 2. \]

72. By Equation 9, the distance is \( D = \frac{|1(-6) - 2(3) - 4(5) - 8|}{\sqrt{1^2 + (-2)^2 + (-4)^2}} = \frac{|-40|}{\sqrt{21}} = \frac{40}{\sqrt{21}} \).
78. First notice that if two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\mathbf{v}_1 = (1, 6, 2)$ and $\mathbf{v}_2 = (2, 15, 6)$, the direction vectors of the two lines respectively. Thus set $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (36 - 30, 4 - 6, 15 - 12) = (6, -2, 3)$. Setting $t = 0$ and $s = 0$ gives the points $(1, 1, 0)$ and $(1, 5, -2)$. So in the notation of Equation 8, $6 - 2 + 0 + d_3 = 0 \Rightarrow d_3 = -4$ and $6 - 10 - 6 + d_2 = 0 \Rightarrow d_2 = 10$.

Then by Exercise 75, the distance between the two skew lines is given by $D = \frac{|-4 - 10|}{\sqrt{36 + 4 + 9}} = \frac{14}{7} = 2$.

Alternate solution (without reference to planes): We already know that the direction vectors of the two lines are $\mathbf{v}_1 = (1, 6, 2)$ and $\mathbf{v}_2 = (2, 15, 6)$. Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = (6, -2, 3)$ is perpendicular to both lines. Pick any point on each of the lines, say $(1, 1, 0)$ and $(1, 5, -2)$, and form the vector $\mathbf{b} = (0, 4, -2)$ connecting the two points. Then the distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is,

$$D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{1}{\sqrt{36 + 4 + 9}} |0 - 8 - 6| = \frac{14}{7} = 2.$$