2.3

10: \[ \begin{bmatrix} 0 & 1 \end{bmatrix} \]

14: \( BC = \begin{bmatrix} 14 & 8 & 2 \end{bmatrix}, BD = \begin{bmatrix} 6 \end{bmatrix}, CD = \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix}, DB = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \), \( DE = \begin{bmatrix} 5 & 5 \end{bmatrix} \)

\[ \begin{bmatrix} 5 & 10 & 15 \end{bmatrix}, AA = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, CC = \begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}, EE = [25] \]

20: All matrices of the form \( \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \). We can write \( A = 2 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 3 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \).

A matrix commutes with \( A \) if and only if it commutes with \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \). It is readily seen that matrices commuting with \( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) are precisely of the above form.

\[ A^n = \begin{cases} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & \text{if } n \equiv 1 (\text{mod} 4) \\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} & \text{if } n \equiv 2 (\text{mod} 4) \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \text{if } n \equiv 3 (\text{mod} 4) \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } n \equiv 0 (\text{mod} 4) \end{cases} \]

2.4

4: The row echelon form of the matrix is \( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \). Hence it is invertible. The inverse is \( \begin{bmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix} \).

12: The row echelon form is \( \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \). Hence \( A \) is invertible. The inverse is

\( \begin{bmatrix} 5 & -20 & -2 & -7 \\ -1 & 0 & 0 & 0 \\ -2 & 6 & 1 & 2 \\ 0 & 3 & 0 & 1 \end{bmatrix} \).
The transformation matrix is \[
\begin{pmatrix}
1 & 3 & 3 \\
1 & 4 & 8 \\
2 & 7 & 12
\end{pmatrix}
\]. It is invertible and the inverse is \[
\begin{pmatrix}
-8 & -15 & 12 \\
4 & 6 & -5 \\
-1 & -1 & 1
\end{pmatrix}
\].

Observe that the difference between any two consecutive rows of $M_n$ is constant. Exploiting this one could make, through a sequence of row operations all but first two rows equal to zero. Hence the rank of $M_n$ for $n \geq 2$ is 2. $M_n$ is invertible only when $n = 1, 2$.

Suppose (without loss of generality) that $n < m$. Then row echelon form of $A$ must of one or more columns consisting of only zeros. Hence $Ax = 0$ will have infinitely many solutions. On the other hand, if $A$ has an inverse (say $B$), there can be only one solution:
\[
A\tilde{x} = \vec{0} \implies BA\tilde{x} = B\vec{0} \implies \tilde{x} = 0.
\]

If $AB$ has inverse $C$ then $(AB)C = C(AB) = I_n$. Since matrix product is associative we can conclude $A(BC) = I_n$ and $(CA)B = I_n$. Hence both $A$ and $B$ are invertible.

The row echelon form of $A$ is \[
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]. $A\tilde{x} = \vec{0}$ if and only if $x_4 = \begin{pmatrix}
-1 \\
-2 \\
1
\end{pmatrix}$.

All columns of the matrix are scalar multiples of \[
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\]. This vector alone spans the image of $A$.

All columns must be multiples of \[
\begin{pmatrix}
7 \\
6 \\
5
\end{pmatrix}
\]. For example, \[
\begin{pmatrix}
7 & 0 & 0 \\
6 & 0 & 0 \\
5 & 0 & 0
\end{pmatrix}
\].

One way to construct such an example is to choose first two columns arbitrarily (say $\vec{v}_1$ and $\vec{v}_2$) but so that neither is a scalar multiple of the other and then choose the third column to be $\vec{v}_3 = \frac{\vec{v}_1 - \vec{v}_2}{2}$. For example, \[
\begin{pmatrix}
1 & 1 & 0 \\
3 & 1 & 1 \\
1 & 1 & 0
\end{pmatrix}
\].

The system is consistent if and only if $y_1 - 3y_3 + 2y_4 = 0$ and $y_2 - 2y_3 + y_4 = 0$. These equations describe the image of $A$. Ans: \[
\begin{pmatrix}
1 & 0 & -3 & 2 \\
0 & 1 & -2 & 1
\end{pmatrix}
\].

The row operations do not change the solutions of $A\tilde{x} = 0$. Hence the kernel of a matrix is unaltered when it is reduced to its row echelon form. However row operations do not preserve the image.