7.4

6: The characteristic equation is \((2 - \lambda)^2 = 0\), so the only eigenvalue is 2, with algebraic multiplicity 2. However, the kernel of \((A-2I)\) is spanned by \[
\begin{bmatrix}
0 \\
1
\end{bmatrix},
\]
so it has dimension 1. Therefore there is no eigenbasis of \(A\) and it is not diagonalizable.

16: The characteristic equation is \((\lambda - 3)(\lambda - 2)(\lambda - 1) = 0\). The eigenspaces for each eigenvalue are spanned by:

\[
v_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\]

If we let \(S = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\), then \(S^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}\). So we have:

\[
D = S^{-1}AS = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]
Note that the diagonal of \(D\) consists of the eigenvalues of \(A\).

18: The characteristic equation is \((\lambda)(\lambda - 1)(\lambda - 2) = 0\). The eigenspaces for each eigenvalue are spanned by:

\[
v_0 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\]

If we let \(S = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\), then \(S^{-1} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}\). So we have:

\[
D = S^{-1}AS = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}
\]

26: The characteristic equation is \((\lambda - 1)^2(\lambda - 2) = 0\). The matrix is diagonalizable if the eigenspace of 1 has dimension 2. So the kernel of \[
\begin{bmatrix}
a & b \\
0 & 1 & c \\
0 & 0 & 0
\end{bmatrix}
\]
must have dimension 2. This happens when \(b = ac\).

30: The characteristic equation is \(\lambda^3 - 3\lambda = a\). By setting \(f(\lambda) = \lambda^3 - 3\lambda\) and following the same procedure as in 7.2, problem 32 (HW9), we see that:
If \( a \in (-2,2) \), there are three distinct eigenvalues, so the eigenvectors of these eigenvalues form an eigenbasis that diagonalize \( A \).

If \( a = 2 \) or \( a = -2 \), there is one eigenvalue with algebraic multiplicity 1 and one with algebraic multiplicity 2. The one with multiplicity 2 is either -1 or 1, and we must check its geometric multiplicity. It turns out to be 1, so there is no eigenbasis and \( A \) is not diagonalizable.

If \( a \in (-\infty,-2) \cup (2,\infty) \), then there is only one real eigenvalue with algebraic multiplicity 1, so \( A \) is not diagonalizable.

58: a) Let \( v \in \operatorname{Im}(A) \), then \( v = Aw \) for some \( w \). \( Av = A^2w = 0 \). Since \( v \) is mapped to 0, \( v \in \ker A \).

b) By the rank-nullity theorem, we know that \( \dim(\operatorname{Im} A)+\dim(\ker A) = 3 \). Since \( \operatorname{Im} A \) is a subspace of \( \ker A \), \( \dim(\operatorname{Im} A) \leq \dim(\ker A) \). So the two options are \( \dim(\operatorname{Im} A) = 1 \) and \( \dim(\operatorname{Im} A)=0 \). But, since \( A \) is nonzero, \( \dim(\operatorname{Im} A) \neq 0 \). Therefore, \( \dim(\operatorname{Im} A)=1 \) and \( \dim(\ker A)=2 \).

c) The problem tells us that \( v_1 \) and \( v_3 \) are linearly independent. Then assume that \( v_2 = av_1 + bv_3 \), a linear combination of \( v_1 \) and \( v_3 \). Then \( Av_2 = aAv_1 + bAv_3 = 0 \), since \( v_1, v_3 \in \ker(A) \). However, \( Av_2 = v_1 \) and this contradicts the assumption that \( v_1 \neq 0 \). This means that \( v_1, v_2, v_3 \) must be linearly independent, so they form a basis for \( \mathbb{R}^3 \).

d) Since \( Av_1 = 0, Av_2 = v_1, Av_3 = 0 \), we have:

\[
B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

7.5

24: The characteristic equation is \( (\lambda-1)(\lambda^2-2\lambda+5) = 0 \). The real eigenvalue is 1, and the complex ones are the solutions to the quadratic equation: \( \lambda = 1 \pm 2i \).

9.1

26: The characteristic equation is \( (\lambda+2)(\lambda-3) = 0 \). The eigenspace of -2 is spanned by \( v_{-2} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \), and the eigenspace of 3 is spanned by \( v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). The expansion of \( x(0) \) in terms of these eigenvectors is:

\[
x(0) = \begin{bmatrix} 7 \\ 2 \end{bmatrix} = -\begin{bmatrix} -2 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

Then the solution is:

\[
x(t) = -e^{-2t} \begin{bmatrix} -2 \\ 3 \end{bmatrix} + 5e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

28: The characteristic equation is \( (\lambda-10)(\lambda-2) = 0 \). The eigenspace of 2 is spanned by \( v_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \), and the eigenspace of 10 is spanned by \( v_{10} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). The expansion of \( x(0) \) in terms of these eigenvectors is:

\[
x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\frac{1}{8} \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \frac{5}{8} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]
Then the solution is:

\[ x(t) = -\frac{1}{8}e^{2t} \begin{bmatrix} -3 \\ 2 \end{bmatrix} + \frac{5}{8}e^{10t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \]

30: The characteristic equation is \( \lambda(\lambda - 5) = 0 \). The eigenspace of 0 is spanned by \( v_0 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \), and the eigenspace of 5 is spanned by \( v_5 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). The initial state of the system is the eigenvector \( \begin{bmatrix} 2 \\ -1 \end{bmatrix} \), so its time evolution is:

\[ x(t) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \]

I.e. it is constant.