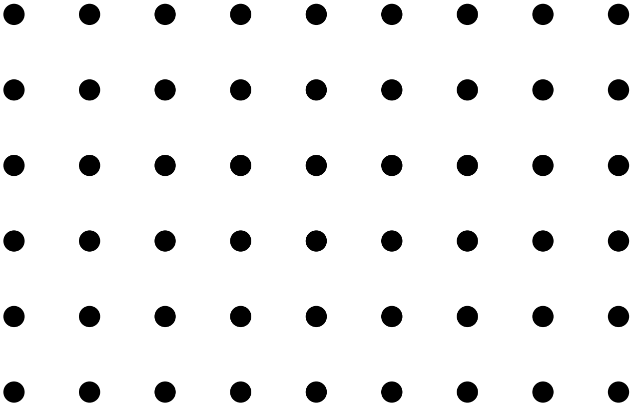
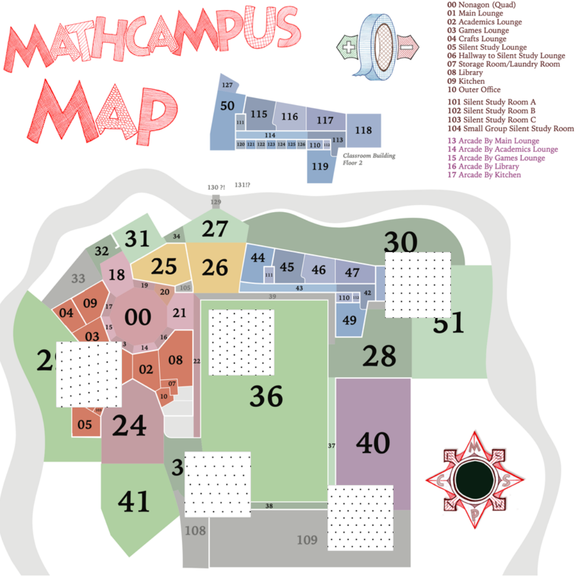
Lattices

- A *lattice* in the plane is an infinitely repeating grid of points in the plane containing the origin.



An atlas of lattices?



00 Nonagon (Quad)
01 Main Lounge
02 Academics Lounge
03 Games Lounge
04 Crafts Lounge
05 Silent Study Lounge
06 Hallway to Silent Study Lounge
07 Storage Room/Laundry Room
08 Library
09 Kitchen
10 Outer Office

101 Silent Study Room A
102 Silent Study Room B
103 Silent Study Room C
104 Small Group Silent Study Room

13 Arcade By Main Lounge
14 Arcade By Academics Lounge
15 Arcade By Games Lounge
16 Arcade By Library
17 Arcade By Kitchen

18 Kitchen Deck
19 Arcade By Assembly Door
20 Arcade By Assembly Stairs
21 Deck Between Quad and Field
22 Walk by Dorm Edge
24 Front Lawn

25 Assembly Hall
26 Amphitheatre

27 Amphitheatre Stage
28 Formal Garden
29 Side Meadow
30 Back Meadow
51 Far Meadow
31 Lower Garden
32 Middle Garden
33 [*coming soon*]

34 Stairs from Garden to Amphitheatre
35 Swing Lawn by Staff Deck
36 Frisbee Field
37 Gym Arcade By Basketball Court
38 Gym Arcade By Gym Door
39 [*coming soon*]

40 Basketball Court
41 President's House

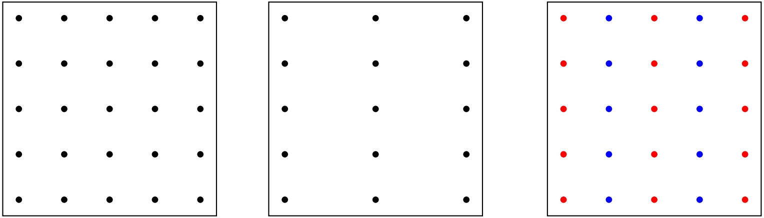
42 Classroom Bldg. First Floor Lobby
43 First Floor Hallway
44 Arch (1A)
45 Douglas (1B)
46 Oxbow (1C)
47 Mint (1D)
48 Union (1E)
49 Ngo (1F)
50 Classroom Building Balcony
110 Front Stairs
111 Back Stairs
112 Elevator
113 Second Floor Lobby
114 Second Floor Hallway
115 Subalpine (2B)
116 Georgia (2C)
117 Peru (2D)
118 Rhode Island (2E)
119 Canyonland (2F)
120 Study Room Do
121 Study Room Re
122 Study Room Mi
123 Study Room Fa
124 Study Room So
125 Study Room La
126 Study Room Ti
127 Stairs from Balcony to Stage
128 [*coming soon*]

105 [*coming soon*]
107 [*coming soon*]
108 [*coming soon*]
109 [*coming soon*]

129-140 [*coming soon*]

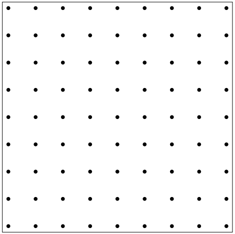
Relations between lattices

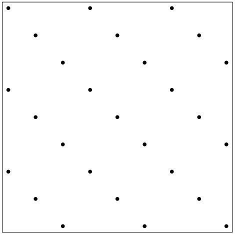
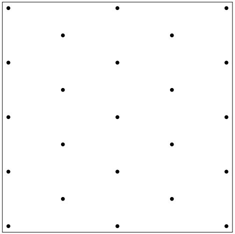
- Let Λ_1, Λ_2 be lattices satisfying $\Lambda_2 \subseteq \Lambda_1$.
- Then Λ_1 is a union of finitely many translates of Λ_2.
- The number of translates is called the *index* of Λ_2 in Λ_1, denoted $[\Lambda_1 : \Lambda_2]$.
- In the example below, $[\Lambda_1 : \Lambda_2] = 2$.



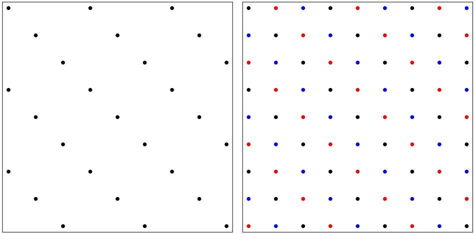
Λ_1 $\quad$ Λ_2 $\quad$ translates of Λ_2 in Λ_1

Exercise: computing the index

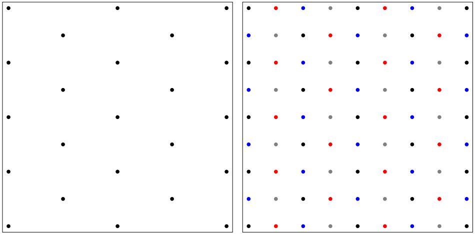
$\Lambda_1 =$ 

$\Lambda_2 =$  or $\Lambda_2 =$ 

Exercise: computing the index



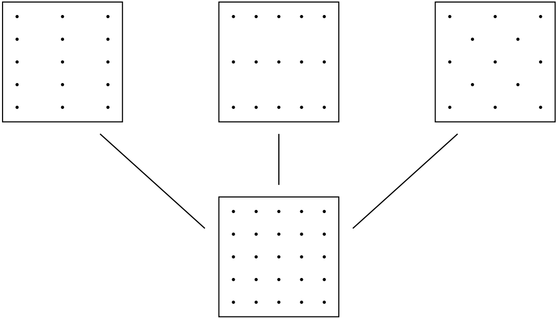
$[\Lambda_1 : \Lambda_2] = 3$



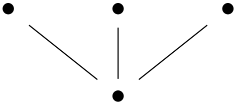
$[\Lambda_1 : \Lambda_2] = 4$

Relations between lattices

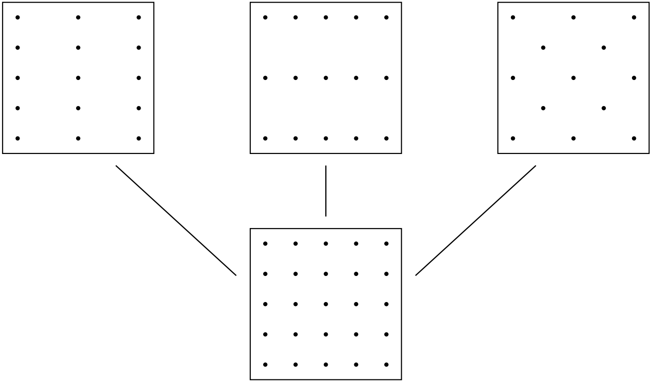
- For any lattice Λ_1, there are exactly three lattices Λ_2 satisfying $\Lambda_2 \subset \Lambda_1$ and $[\Lambda_1 : \Lambda_2] = 2$.



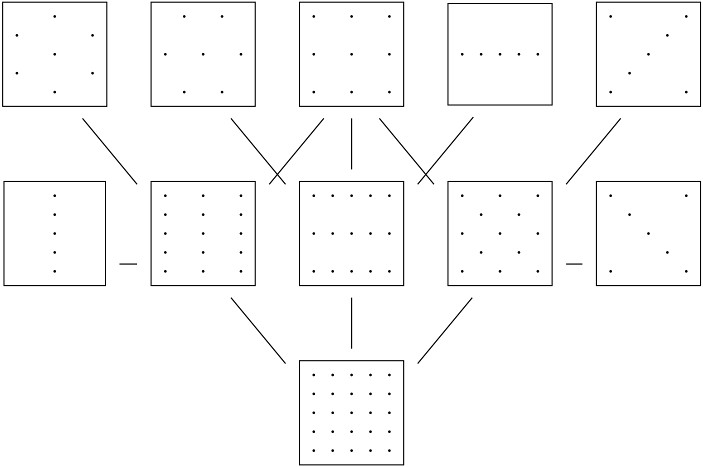
- We can summarize this information in a graph.



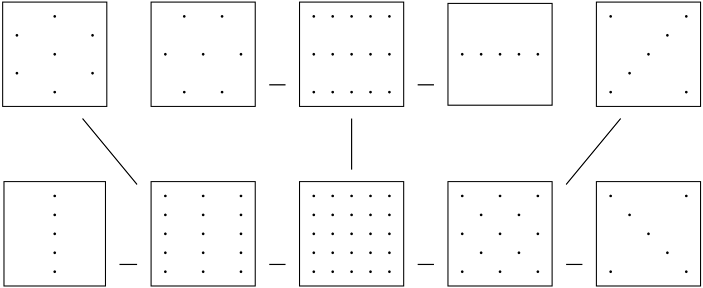
- We construct a graph as follows:
  - Each lattice in the plane is a vertex of the graph.
  - We draw an edge between the lattices Λ_1 and Λ_2 if $\Lambda_2 \subset \Lambda_1$ and $[\Lambda_1 : \Lambda_2] = 2$.
- We just saw a small piece of the graph:



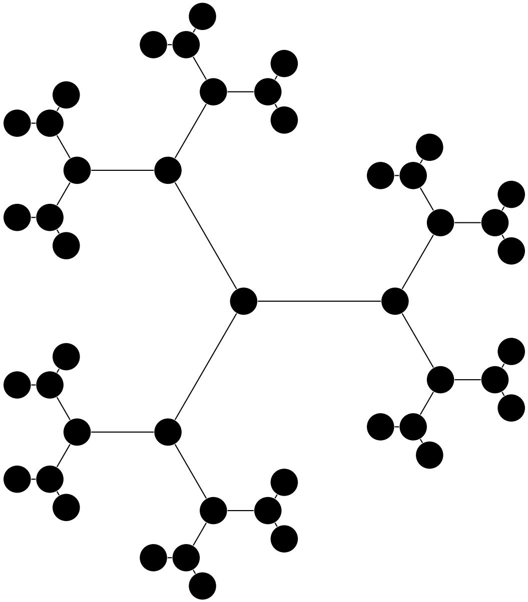
- A slightly larger piece of the graph:



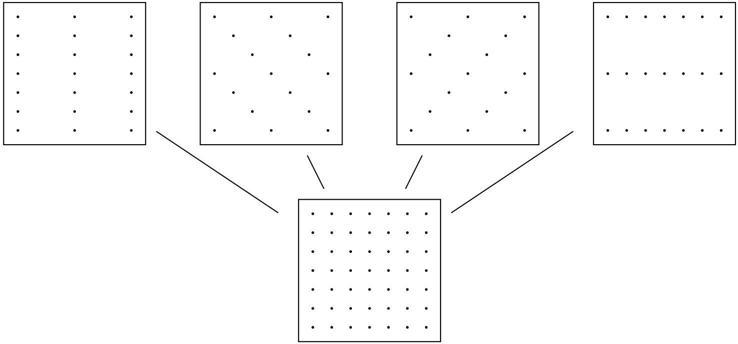
- Declare two vertices to be *equivalent* if their lattices are related by scaling.



The new graph is an infinite 3-regular tree, called a *Bruhat–Tits tree*.

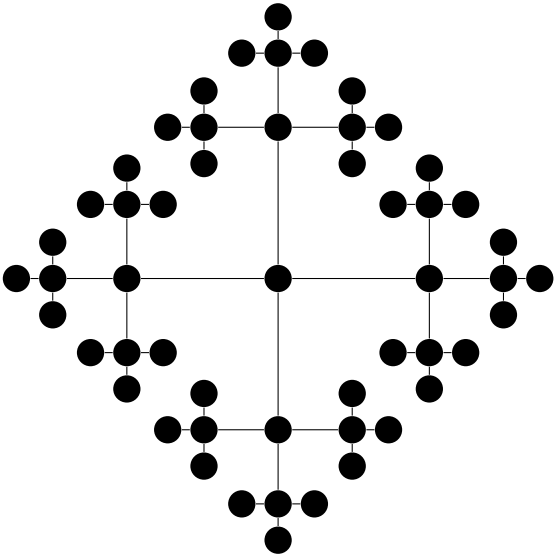


- Suppose that we instead consider lattices Λ_1, Λ_2 satisfying $[\Lambda_1 : \Lambda_2] = 3$.



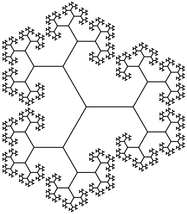
- We draw a graph according to the same procedure as before:
  - Each lattice in the plane is a vertex of the graph.
  - Draw an edge between the lattices Λ_1 and Λ_2 if $\Lambda_2 \subset \Lambda_1$ and $[\Lambda_1 : \Lambda_2] = 3$.
  - Then identify vertices related by scaling of lattices.

- This time, we get an infinite 4-regular tree:

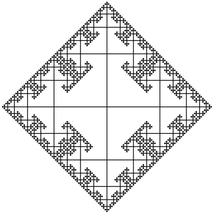


A nice pattern

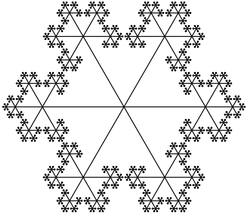
- Suppse we draw an edge between lattices satisfying $[\Lambda_2 : \Lambda_1] = p$:



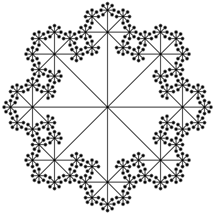
$p = 2$



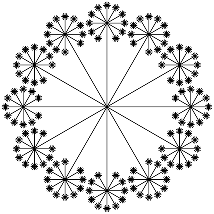
$p = 3$



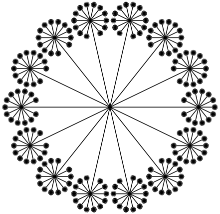
$p = 5$



$p = 7$

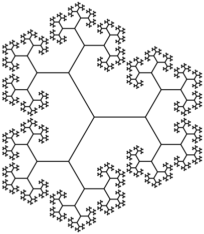


$p = 11$

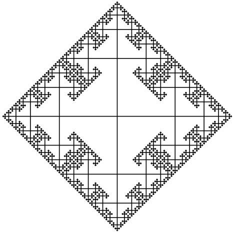


$p = 13$

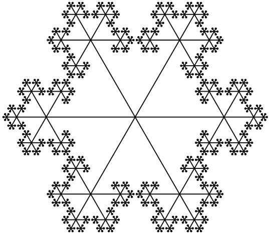
- For any prime p, if we draw an edge between lattices satisfying $[\Lambda_2 : \Lambda_1] = p$, we get an infinite $p+1$-regular tree.



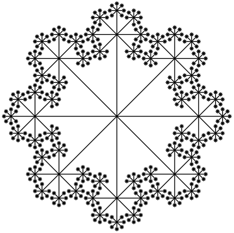
$p = 2$



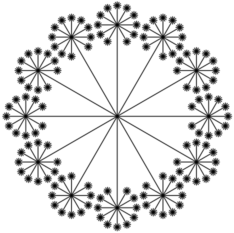
$p = 3$



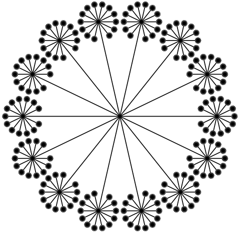
$p = 5$



$p = 7$

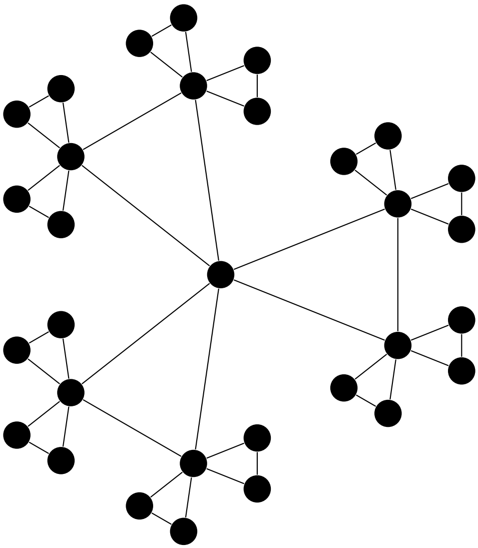


$p = 11$



$p = 13$

If we draw an edge between lattices satisfying $[\Lambda_1 : \Lambda_2] = 4$ for example, we no longer get a tree:



Generalizations of this setup

- Higher-dimensional lattices
- Lattices with symmetry

Three-dimensional lattices

- Construct a graph as follows:
  - Vertices are lattices in three-dimensional space.
  - Draw an edge between Λ_1 and Λ_2 if $\Lambda_2 \subset \Lambda_1$ and $[\Lambda_1 : \Lambda_2] = 2$.
- Then identify two vertices if their lattices are related by scaling.
- The resulting graph is called a *building*.

A building

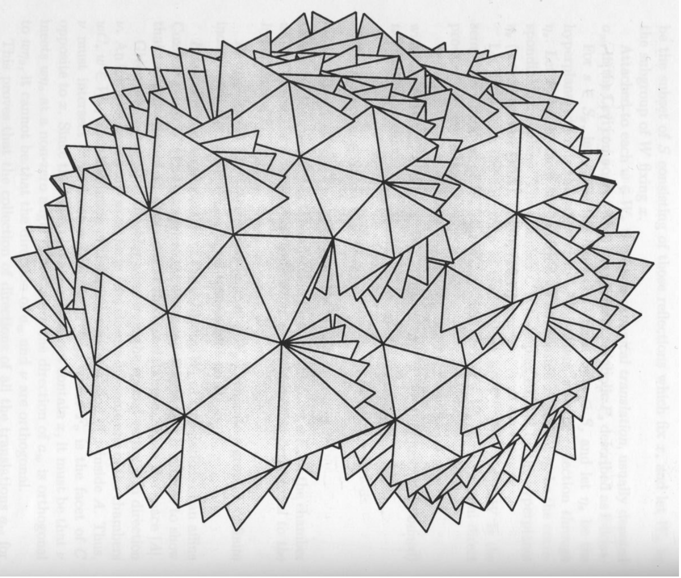
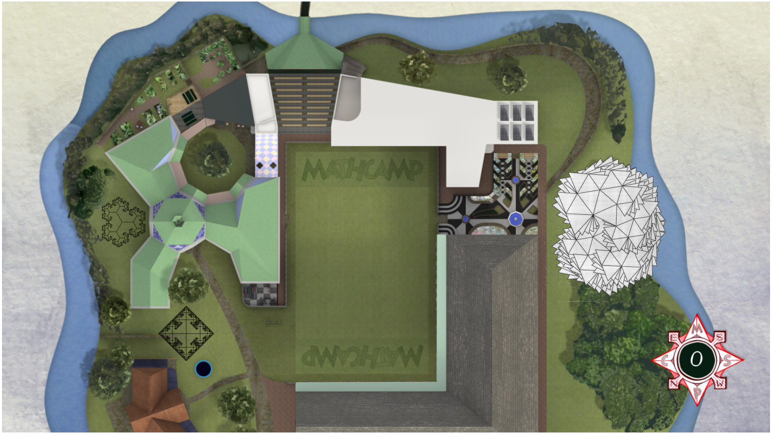


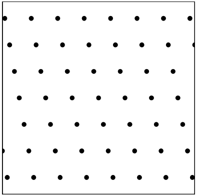
Image credit: P. Garrett, *Buildings and Classical Groups*

Mathcampus p-adic expansion plan

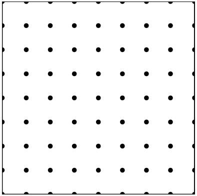


Symmetries of lattices in two dimensions

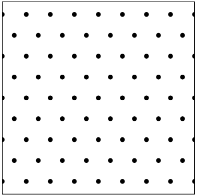
- All lattices have twofold rotational symmetry.
- In the plane, some lattices also have fourfold or sixfold rotational symmetry.



Twofold symmetry

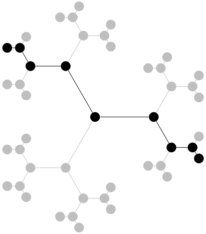


Fourfold symmetry

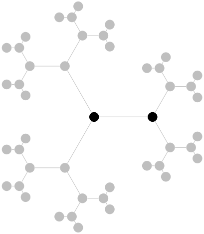


Sixfold symmetry

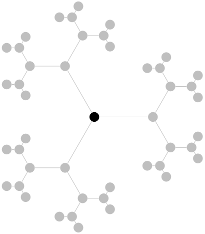
- Which lattices have more than twofold symmetry?
- On the Bruhat–Tits tree, the set of vertices corresponding to lattices with fourfold or sixfold symmetry is either:



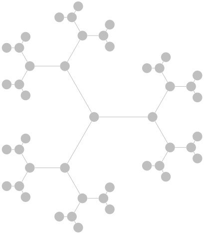
an infinite path,



two adjacent vertices,



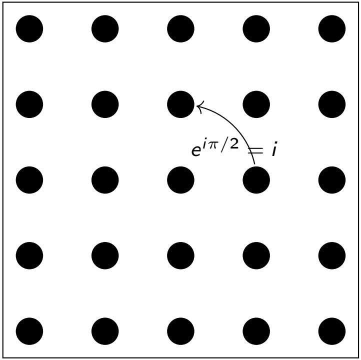
a single vertex, or

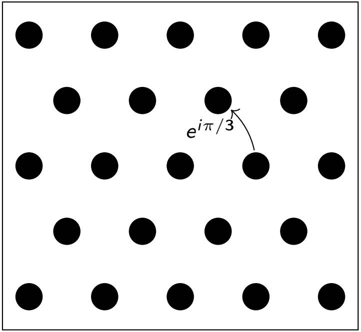


empty

Symmetries and complex numbers

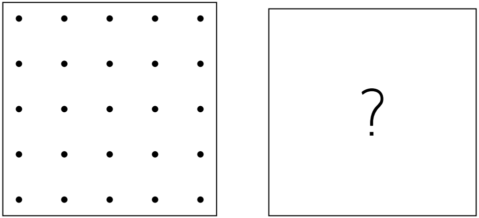
- Before moving to higher dimensions, let's consider how to express symmetries using complex numbers.
- In the complex plane, multiplication by $e^{i\theta}$ corresponds to rotation by θ.





Symmetries and complex numbers

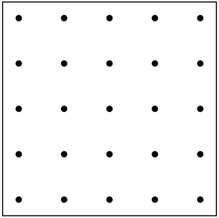
- A lattice Λ has fourfold rotational symmetry if and only if $i\Lambda = \Lambda$.
- If $i\Lambda = \Lambda$, then $(1+i)\Lambda \subset \Lambda$, since the sum of two elements of Λ is also in Λ.
- Exercise: draw $(1+i)\Lambda$.



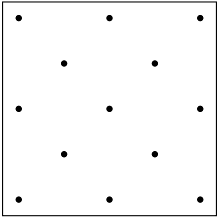
Λ $\qquad$ $(1+i)\Lambda$

Symmetries and complex numbers

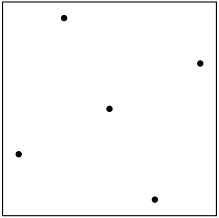
- A lattice Λ has fourfold rotational symmetry if and only if $i\Lambda = \Lambda$.
- If $i\Lambda = \Lambda$, then $(1+i)\Lambda \subset \Lambda$, since the sum of two elements of Λ is also in Λ. Similarly, $(a+bi)\Lambda \subset \Lambda$ for all $a, b \in \mathbb{Z}$.



Λ

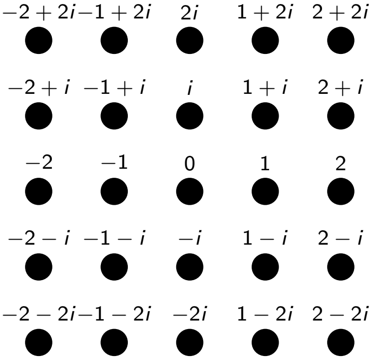


$(1+i)\Lambda$



$(2+i)\Lambda$

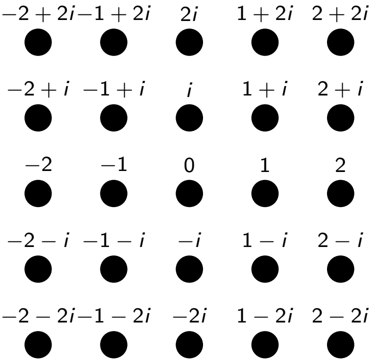
- Let $\mathbb{Z}[i] := \{a + bi | a, b \in \mathbb{Z}\}$.



- A lattice Λ has fourfold symmetry if and only if

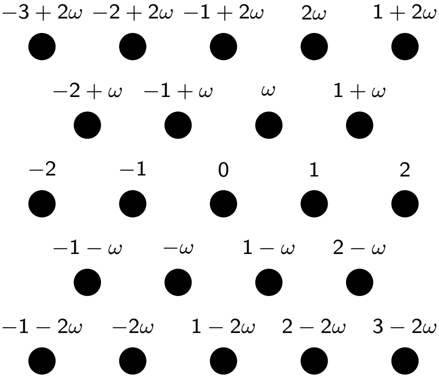
$$\{z \in \mathbb{C} | z\Lambda \subseteq \Lambda\} = \mathbb{Z}[i] .$$

- If this condition is satified, we say that Λ is a $\mathbb{Z}[i]$-*ideal*.



- $\mathbb{Z}[i]$ is an example of an *order*: in addition to being a lattice, it satisfies
  - $1 \in \mathbb{Z}[i]$.
  - For all $z_1, z_2 \in \mathbb{Z}[i]$, $z_1 z_2 \in \mathbb{Z}[i]$.

- Let $\mathbb{Z}[\omega] := \{a + b\omega | a, b \in \mathbb{Z}\}$, where $\omega = e^{\pi i/3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$.



- $\mathbb{Z}[\omega]$ is also an order:
  - $1 \in \mathbb{Z}[\omega]$.
  - For all $z_1, z_2 \in \mathbb{Z}[\omega]$, $z_1 z_2 \in \mathbb{Z}[\omega]$.
- A lattice Λ has sixfold rotational symmetry if and only if

$$\{z \in \mathbb{C} | z\Lambda \subseteq \Lambda\} = \mathbb{Z}[\omega] .$$

- For any lattice $\Lambda \subset \mathbb{C}$, define

$$\mathrm{End}(\Lambda) := \{z \in \mathbb{C} | z\Lambda \subseteq \Lambda\} \ .$$

- For most lattices Λ, $\mathrm{End}(\Lambda) = \mathbb{Z}$. But $\mathrm{End}(\Lambda)$ can be an order such as $\mathbb{Z}[i]$ or $\mathbb{Z}[\omega]$, as we just saw.
- If $\mathrm{End}(\Lambda)$ is an order (i.e. if $\mathrm{End}(\Lambda)$ is a lattice), then we say that Λ is an $\mathrm{End}(\Lambda)$-*ideal*.

- Our construction of Ramanujan graphs will involve lattices in four dimensions with symmetry.
- We will need a four-dimensional replacement for the complex numbers.
- We will use the *quaternions*.

Quaternions

- Define the set of *quaternions* by

$$\mathbb{H} := \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\} .$$

- Multiplication of quaternions is defined by the rules

$$i^2 = j^2 = k^2 = -1 ,$$

$$ij = -ji = k, \quad ki = -ik = j, \quad jk = -kj = i .$$

Orders and ideals in $\mathbb{H}$

- An *order* in $\mathbb{H}$ is a lattice $\mathcal{O} \subset \mathbb{H}$ such that:
  - $1 \in \mathcal{O}$.
  - For all $z_1, z_2 \in \mathcal{O}$, $z_1 z_2 \in \mathcal{O}$.
- Let Λ be a lattice in $\mathbb{H}$, and let $\mathcal{O}$ be an order in $\mathbb{H}$. We say that Λ is a *left $\mathcal{O}$-ideal* if

$$\{z \in \mathbb{H} | z\Lambda \subseteq \Lambda\} = \mathcal{O}\,.$$

Exercise: orders in $\mathbb{H}$

- An *order* in $\mathbb{H}$ is a lattice $\mathcal{O} \subset \mathbb{H}$ such that:
  - $1 \in \mathcal{O}$.
  - For all $z_1, z_2 \in \mathcal{O}$, $z_1 z_2 \in \mathcal{O}$.
- Which of these are orders in $\mathbb{H}$?

$$\mathbb{Z}$$

$$\{a + bi + cj + dk | a, b, c, d \in \mathbb{Z}\}$$

$$\{a + bi + cj + dk | a, b, c, d \in 2\mathbb{Z}\}$$

$$\{a + bi + cj + dk | a \in \mathbb{Z},\ b, c, d \in 2\mathbb{Z}\}$$

$$\{a + bi + cj + dk | a, b, c, d \in \frac{1}{2}\mathbb{Z}\}$$

Exercise: orders in $\mathbb{H}$

- Which of these are orders?

$$\mathbb{Z} \text{ No}$$

$\mathbb{Z}$ is not a lattice in $\mathbb{H}$.

$$\{a + bi + cj + dk | a, b, c, d \in \mathbb{Z}\} \text{ Yes}$$

$$\{a + bi + cj + dk | a, b, c, d \in 2\mathbb{Z}\} \text{ No}$$

Does not contain 1.

$$\{a + bi + cj + dk | a \in \mathbb{Z},\ b, c, d \in 2\mathbb{Z}\} \text{ Yes}$$

$$\{a + bi + cj + dk | a, b, c, d \in \frac{1}{2}\mathbb{Z}\} \text{ No}$$

$1/2$ is in the lattice but $(1/2)^2$ is not.

An order in $\mathbb{H}$

- Consider the lattice $\mathcal{O}$ in $\mathbb{H}$ generated by

$$1, \quad \frac{i-\sqrt{3}k}{2}, \quad i-\sqrt{3}j, \quad \frac{1+3i+\sqrt{3}j+\sqrt{3}k}{2}\,.$$

- $\mathcal{O}$ is an order: $1 \in \mathcal{O}$ and for all $z_1, z_2 \in \mathcal{O}$, $z_1 z_2 \in \mathcal{O}$. For example, we can check that

$$\begin{aligned}
&\left(\frac{i-\sqrt{3}k}{2}\right)\left(i-\sqrt{3}j\right)\\
=&\frac{1}{2}\left(i\cdot i - i\cdot\sqrt{3}j - \sqrt{3}k\cdot i + \sqrt{3}k\cdot\sqrt{3}j\right)\\
=&\frac{1}{2}\left(-1-\sqrt{3}k-\sqrt{3}j-3i\right)\\
=&-\frac{1+3i+\sqrt{3}j+\sqrt{3}k}{2}\in\mathcal{O}\,.
\end{aligned}$$

- We will describe a procedure that constructs a graph given:
  - An order $\mathcal{O} \subset \mathbb{H}$.
  - A prime p.
- It will turn out that this graph is *usually* Ramanujan.
- More precisely, for any fixed $\mathcal{O}$, the graph is Ramanujan for all but finitely many p.
- We will show how to construct the Ramanujan graph from the first lecture using this procedure.