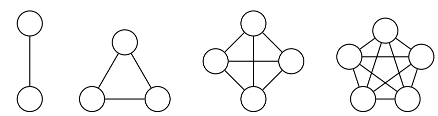
Ramanujan Graphs, Quaternions, and Number Theory homework - Day 1

Some of these exercises are rather time-consuming/difficult/open-ended. You don't need to do all of them; just choose the ones that look the most interesting to you.

1. Go to `https://www.sagemath.org/` and either install Sage or set up a CoCalc account. Download the example notebook at `https://people.maths.ox.ac.uk/gulotta/Ramanujan.ipynb` and open it in Sage. Then try generating some graphs of your own. The Sage documentation for Brandt modules is at `https://doc.sagemath.org/html/en/reference/modmisc/sage/modular/quatalg/brandt.html`.

2. A *complete graph* is a graph where every pair of vertices is connected by an edge. The complete graph with n vertices is denoted K_n.

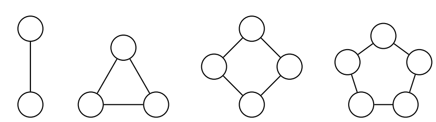
   

   K_2 K_3 K_4 K_5

   (a) Write down the adjacency matrices of K_2 and K_3. What are their eigenvalues?

   (b) Find the eigenvalues of the adjacency matrix of K_n. (Hint: Let A_n be the adjacency matrix. First find the eigenvalues of $A_n + I_n$, where I_n is the $n \times n$ identity matrix.)

3. (a) Consider the *cycle graph* C_n.

   C_2 C_3 C_4 C_5

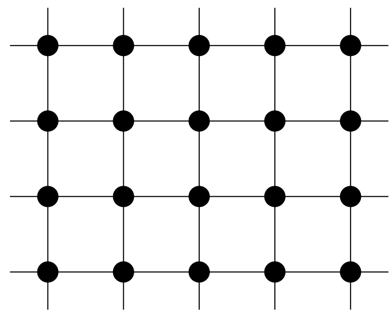
   Verify that for each integer m, the vectors

   $$(\cos(2\pi \cdot 0 \cdot m/n), \cos(2\pi \cdot 1 \cdot m/n), \ldots, \cos(2\pi \cdot (n-1) \cdot m/n))$$

   $$(\sin(2\pi \cdot 0 \cdot m/n), \sin(2\pi \cdot 1 \cdot m/n), \ldots, \sin(2\pi \cdot (n-1) \cdot m/n))$$

   are eigenvectors of the adjacency matrix of C_n, with eigenvalue $2\cos(2\pi m/n)$.

   (b) Explain why the eigenvalues of the adjacency matrix of a $n_1 \times n_2$ wrapping square grid are of the form $2\cos(2\pi m_1/n_1) + 2\cos(2\pi m_2/n_2)$ for integer m_1, m_2.



4. Write a computer program that generates random regular graphs. How large are the eigenvalues of the adjacency matrices typically?

5. Write a computer program that generates regular graphs of varying sizes without using randomness. How large the eigenvalues of the adjacency matrices?

6. Read the article "On the second eigenvalue of a graph" by A. Nilli, which proves Proposition 1.8 in the notes. It should be available for free online.

7. (if you know some group theory) The computation in this exercise will be useful for tomorrow's class. How many subgroups of index n does $\mathbb{Z}^2$ have

   (a) when $n = 2$?

   (b) when $n = 3$, 4, or 5?

   (c) when n is prime?

   (d) in general?

   (Hint: a subgroup of $\mathbb{Z}^2$ of index n must contain $n\mathbb{Z}^2$. So there is a bijection between subgroups of index n in $\mathbb{Z}^2$ and subgroups of index n in $(\mathbb{Z}/n\mathbb{Z})^2$.)