2.3.X1 Let $p$ be a prime congruent to 1 mod 3.

(a) Show that the equation $t^2 + t + 1 \equiv 0 \pmod{p}$ has a solution. (Hint: since $t^2 + t + 1 = \frac{t^3 - 1}{t - 1}$, it suffices to show that $t^3 - 1 \equiv 0 \pmod{p}$ has a solution other than $t \equiv 1 \pmod{p}$. To show that such a solution exists, consider the set $S = \{(a, b, c) \mid a, b, c \in \{1, 2, \ldots, p - 1\}, abc \equiv 1 \pmod{p}\}$. Show that the number of elements of $S$ is a multiple of 3, and that the number of elements of $S$ with $a, b, c$ not all the same is also a multiple of 3. Conclude the number of elements of $S$ with $a = b = c$ must be a multiple of 3, and therefore cannot be exactly 1.)

(b) Now let $t$ be an integer satisfying $t^2 + t + 1 \equiv 0 \pmod{p}$. Let $\Lambda$ be the lattice generated by the vectors $(p, 0)$ and $(t, 1)$. Let $C$ be the disc $x^2 + xy + y^2 < 2p$. Apply Minkowski’s theorem to $\Lambda$ and $C$ to show that $p = a^2 + ab + b^2$ for some integers $a, b$.

2.3.X2 Let $p$ be a prime. Suppose that the equation $t^2 + t + 5 \equiv 0 \pmod{p}$ has a solution. Show that $p = a^2 + ab + 5b^2$ for some integers $a, b$. (Hint: Can $2p$ be of the form $a^2 + ab + 5b^2$?)

If you replace 5 with 11