Kleinian Groups and Fractals homework - Day 4

1. Play around with the fractal generator at `https://guests.mpim-bonn.mpg.de/gulotta/kleinian/`. Do you notice any interesting patterns? If so, show me at TAU or send me a message on Slack and we might discuss it during tomorrow's class!

   If you're feeling ambitious, you can also modify the fractal generator (source code at `https://github.com/dgulotta/kleinian`) or write your own.

2. Given two intersecting circles, we can measure the angle at which they intersect. In Exercise 2 of the Day 2 homework, we showed that Möbius transformations preserve circles and angles. In this exercise, we will show that there is a notion of "distance" between nonintersecting circles that is preserved by Möbius transformations.

   In Exercise 3 of the Day 3 homework, we described a way of representing a circle by a Hermitian matrix with negative determinant. Suppose that circles C_1 and C_2 are represented by matrices A_1 and A_2. Since scaling the matrix does not change the circle, we will assume $\det A_1 = \det A_2 = -1$. Recall that if B is a matrix, then the matrix corresponding to the circle BC_i is $(B^{-1})^\dagger A_i B^{-1}$.

   Define
   $$\tau(C_1, C_2) := \left|\operatorname{tr}(A_1 A_2^{-1})\right| .$$

   (a) Recall that the *trace* of a matrix is the sum of its diagonal entries, and that conjugating a matrix does not change its trace. Explain why $\tau(C_1, C_2)$ is invariant under Möbius transformations, i.e. for any invertible matrix B, $\operatorname{tr} A_1 A_2^{-1} = \operatorname{tr}(B^{-1})^\dagger A_1 B^{-1} \left((B^{-1})^\dagger A_2 B^{-1}\right)^{-1}$.

   (b) Show that if C_1 and C_2 intersect at an angle θ, then
   $$\tau(C_1, C_2) = 2\left|\cos\theta\right| .$$

   (c) Show that if C_1 and C_2 are concentric, then
   $$\tau(C_1, C_2) = \frac{r(C_1)}{r(C_2)} + \frac{r(C_2)}{r(C_1)} .$$
   where $r(C_i)$ is the radius of C_i.

   (d) Show that if C_1 and C_2 do not intersect, then
   $$\tau(C_1, C_2) > 2 .$$
   It therefore seems reasonable to call $\cosh^{-1}(\tau(C_1, C_2)/2)$ the "distance" between C_1 and C_2.

(e) Show that if C_1 and C_2 do not intersect and one circle is contained in the interior of the other, then

$$\tau(C_1, C_2) \leq \frac{r(C_1)}{r(C_2)} + \frac{r(C_2)}{r(C_1)}\,.$$

3. As promised on Tuesday, we can now show that the sequences of Schottky discs

$$D_{g_1} \supset D_{g_1 g_2} \supset D_{g_1 g_2 g_3} \supset \cdots \qquad (*)$$

shrink to zero size.

For any $g_1, g_2, \ldots, g_n$, let $C_{g_1 g_2 \cdots g_n}$ denote the boundary of $D_{g_1 g_2 \cdots g_n}$.

(a) Explain why, for any $g_1, g_2, \ldots, g_n, g_{n+1} \in \{a, a^{-1}, b, b^{-1}\}$,

$$\tau(C_{g_1 \cdots g_n}, C_{g_1 \cdots g_n g_{n+1}}) = \tau(C_{g_n}, C_{g_n g_{n+1}})\,.$$

(b) Conclude that

$$\frac{r(C_{g_1 \cdots g_n g_{n+1}})}{r(C_{g_1 \cdots g_n})} + \frac{r(C_{g_1 \cdots g_n})}{r(C_{g_1 \cdots g_n g_{n+1}})} \geq \tau(C_{g_n}, C_{g_n g_{n+1}})\,.$$

(c) Since the function $f(x) = x + 1/x$ is decreasing for $x < 1$, conclude that

$$\frac{r(C_{g_1 \cdots g_n g_{n+1}})}{r(C_{g_1 \cdots g_n})} \leq \lambda\,,$$

where λ satisfies $\lambda + \lambda^{-1} = \tau(C_{g_n}, C_{g_n g_{n+1}})$ and $\lambda < 1$.

(d) Since there are only finitely many possibilities for the pair g_n, g_{n+1}, conclude that the radii of the discs $(*)$ decrease exponentially.