# A SINGULAR ENERGY MINIMIZING FREE BOUNDARY

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Abstract. We consider the problem of minimizing the energy functional $\int(|\nabla u|^2 + \chi_{\{u>0\}})$. We show that the singular axisymmetric critical point of the functional is an energy minimizer in dimension 7. This is the first example of a non-smooth energy minimizer. It is analogous to the Simons cone, a least area hypersurface in dimension 8.

## 1. Introduction

A global minimizer for the functional

$$J(u, B) = \int_B (|\nabla u|^2 + \chi_{\{u>0\}})$$

is a function $u \in H^1_{loc}(\mathbb{R}^n)$ such that, for any ball $B \subset \mathbb{R}^n$ and any function $v \in H^1(B)$, such that $u - v \in H^1_0(B)$, $J(v, B) \geq J(u, B)$. Throughout this paper we restrict ourselves to non-negative functions $u$, which is known as the one-phase case. The free boundary of $u$, $F(u)$ is defined as the boundary of its positive phase $\{u > 0\}$.

Consider the (unique up to scalar multiple) positive harmonic function $Z$ in the cone

$$\Gamma = \left\{ x \in R^n : |x_n| < t_n \sqrt{x_1^2 + ... + x_n^2} \right\}$$

which is 0 on $\partial\Gamma$. For $n \geq 3$, let $t_n > 0$ be the unique constant such that $Z$ is homogeneous of degree 1. $\Gamma$ is connected and its complement consists of two right circular cones with central axes in the directions $\pm e_n$. The inner normal derivative of $Z$, $Z_\nu$, is homogeneous of degree 0. Rotational symmetry implies that $Z_\nu$ is constant. Therefore, one can choose a scalar multiple $c$ so that $cZ_\nu = 1$ on $\partial\Gamma\backslash\{0\}$. Let $U$ be the function which equals $cZ$ in $\Gamma$ and 0 outside of $\Gamma$. It is not hard to show that $U$ is a critical point for the energy functional $J(\cdot, B)$ for every ball $B \subset \mathbb{R}^n$.

**Theorem 1.1.** *In dimension $n = 7$, $U$ is a global energy minimizer for the functional $J(\cdot, B)$.*

A brief history of this problem is as follows. In [1], Alt and Caffarelli proved that in two dimensions, singular points of the free boundary cannot occur. They also developed a partial regularity theory in higher dimensions and they observed that in dimension $n = 3$, the critical point $U$ is not an energy minimizer. G.Weiss [11] showed that there exists a critical dimension $k$, $3 \leq k \leq \infty$, such that the energy minimizing free boundaries are smooth for $n < k$. Moreover, they can have singular points in dimension $n = k$, and in higher dimensions the singularities form a closed

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set of dimension at most $n-k$. Weiss's theorem does not rule out the possibility that $k=+\infty$, but the strong analogy with the theory of minimal surfaces suggests that there should be a finite critical dimension. In [6], Caffarelli, Jerison and Kenig proved that there are no singular free boundary minimizers in dimension $n=3$, thus $k \geq 4$. They also showed that $U$ is not an energy minimizer in dimension $n \leq 6$. Their proof suggests that the critical dimension $k$ at which minimizing free boundaries admit singularities is $k=7$.

The present paper shows that $k \leq 7$ by providing an example of a singular energy minimizing free boundary in dimension $n=7$. This is analogous to the Simons cone

$$S = \{x \in \mathbb{R}^8 | x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\},$$

which is an example of a singular hypersurface of least area in dimension $n=8$. The minimality of the Simons cone was first proved by Bombieri, De Giorgi and Giusti using a calibration argument [3]. Another proof based on maximum principle techniques can be found in [10]. Our proof is closer in spirit to the proof of [10]. It uses families of subsolutions and supersolutions.

For a fixed ball $B \subset \mathbb{R}^n$, centered at the origin, it is well known that an energy minimizer $u$ for $J(\cdot, B)$, among all competitors $v$ which coincide with $U$ on $\partial B$, is a weak solution (see Section 2 for definition) to the one-phase free boundary problem:

$$\begin{cases} \Delta u = 0 & \text{in } B^+(u) = \{x \in B : u(x) > 0\}, \\ u_\nu = 1 & \text{on } \partial B^+(u) \cap B. \end{cases}$$

We construct a family of weak supersolutions and a family of weak subsolutions to the problem above, which approach $U$ respectively from above and below. Using comparison techniques, we show that $u$ must lie between such families. This forces the minimizer $u$ to coincide with $U$. We point out that unlike the case of the Simons cone, the construction of the two families is asymmetric. The subsolution $V$ is a relatively straightforward interior perturbation of $U$, whose positive phase is disjoint from the origin. The supersolution $W$ must have a positive phase that contains a full neighborhood of the origin, which makes the construction much more complicated.

The proofs of the subsolution and supersolution properties depend on numerical calculations, carried out using Mathematica. In the case of the subsolution, the calculation is just evaluation of some classical Legendre functions and derivatives. The supersolution proof is considerably harder, but it boils down to evaluation and bounds in specific finite regions of several such functions.

Thus, in principle the computational part of the proof could be carried out on a calculator. In practice, we first needed to experiment a great deal with special functions and many choices of parameters and functional forms before coming up with the ones that work.

## 2. Definitions and Main tools

Let $\Omega$ be a domain in $\mathbb{R}^n$, and $u$ a real valued function on $\Omega$. Set, $\Omega^+(u) = \{x \in \Omega : u(x) > 0\}$ and $\Omega^- = \{x \in \Omega : u \leq 0\}^\circ$.

We start by defining the notion of weak solution to the one-phase free-boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^+(u), \\ u_\nu = 1 & \text{on } \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{2.1}$$

Denote by $F(u)$ the free boundary $\partial\Omega^+(u) \cap \Omega$.

**Definition 2.1.** Let $u$ be a nonnegative continuous function in $\Omega$. We will say that $u$ is a weak solution to (2.1) in $\Omega$ if and only if the following conditions are satisfied:

(i) $\Delta u = 0$ in $\Omega^+(u)$;

(ii) If $x_0 \in F(u)$ and $F(u)$ has at $x_0$ a one-sided tangent ball (i.e. there exists $B_\epsilon$ such that $x_0 \in \partial B_\epsilon$ and $B_\epsilon$ is contained either in $\Omega^+$ or in $\Omega^-$), then, for $\nu$ the unit radial direction of $\partial B_\epsilon$ at $x_0$ into $\Omega^+(u)$,

$$u(x) = (x - x_0, \nu)^+ + o(|x - x_0|), \text{ as } x \to x_0.$$

**Definition 2.2.** Let $v$ be a nonnegative continuous function in $\Omega$. We will say that $v$ is a weak subsolution (resp. supersolution) to (2.1) in $\Omega$, if and only if the following conditions are satisfied:

(i) $\Delta v \geq 0$ (resp. $\leq 0$) in $\Omega^+(v)$;

(ii) If $x_0 \in F(v)$ and $F(v)$ has at $x_0$ a tangent ball $B_\epsilon$ from the positive (resp. zero) side (i.e. $B_\epsilon \subset \Omega^+(v)$ (resp. $\Omega^-(v)$), $x_0 \in \partial B_\epsilon$), then, for some $\alpha \geq 1$ (resp. $\alpha \leq 1$) and $\nu$ the unit inner (resp. outer) radial direction of $\partial B_\epsilon$ at $x_0$,

$$v(x) = \alpha(x - x_0, \nu)^+ + o(|x - x_0|), \text{ as } x \to x_0.$$

We will say that $u$ is a strict weak subsolution (resp. supersolution) if the constant $\alpha$ in Definition 2.2 is strictly greater (resp. smaller) than 1.

The following comparison result can be found in [4].

**Lemma 2.3.** *Let $v_\rho$, $a \leq \rho \leq b$, be a family of weak subsolutions to* (2.1) *in $\Omega$, continuous in $\overline{\Omega} \times [a, b]$. Let $u$ be a weak solution to* (2.1) *in $\Omega$, continuous in $\overline{\Omega}$. Assume that*

(i) *$v_a \leq u$ in $\Omega$;*

(ii) *$v_\rho \leq u$ on $\partial\Omega$, and $v_\rho < u$ in $[\overline{\Omega^+(v_\rho)} \cap \partial\Omega]$, for all $\rho \in [a, b]$;*

(iii) *at every $x_0 \in F(v_\rho)$, there exists a tangent ball from the positive side;*

(iv) *$\overline{\Omega^+(v_\rho)}$ is continuous (in the Hausdorff metric)in $\rho$.*

*Then $u \geq v_\rho$ in $\Omega$, for any $\rho$.*

We will also need a slightly different comparison result for a family of supersolutions, which can be proved by similar techniques to Lemma 2.3.

**Definition 2.4.** A non-negative function $u \in C(\Omega)$ satisfies condition (C) if for every $x_0 \in \overline{\Omega^+(u)} \cap \partial\Omega$, such that $u$ is zero in a neighborhood of $x_0$ in $\partial\Omega$, and $x_0$ admits an exterior tangent ball, i.e. $x_0 \in \partial B$ with $B \subset \mathbb{R}^n \setminus \Omega^+(u)$, then there exists $\alpha \geq 1$ such that

$$u(x) = \alpha(x - x_0, \nu)^+ + o(|x - x_0|), \text{ as } x \to x_0,$$

with $\nu$ outward unit normal at $\partial B$.

**Lemma 2.5.** *Let $w_\rho$, $a \leq \rho \leq b$, be a family of weak strict supersolutions to* (2.1) *in $\mathbb{R}^n$, continuous in $\mathbb{R}^n \times [a, b]$. Let $u$ be a weak solution to* (2.1) *in $\Omega$, continuous in $\overline{\Omega}$ and satisfying condition* (C)*. Assume that,*

(i) *$u \leq w_a$ in $\Omega$;*

(ii) *$u \leq w_\rho$ on $\partial\Omega$ for any $\rho$, and $w_\rho(x_0) > 0$ at each $x_0 \in \overline{\Omega^+(u)} \cap \partial\Omega$ such that $u$ is not identically zero in any boundary neighborhood of $x_0$;*

(iii) *at every* $x_0 \in F(w_\rho)$, *there exists a tangent ball from the zero side;*
(iv) $\overline{\Omega^+(w_\rho)}$ *is continuous (in the Hausdorff metric) in* $\rho$.

*Then* $u \leq w_\rho$ *in* $\Omega$, *for any* $\rho$.

*Proof.* Define $A = \{\rho \in [a,b] \mid u \leq w_\rho \text{ on } \overline{\Omega}\}$. Hypothesis (i) together with the continuity of $w_\rho$ in $\rho$ imply that $A$ is closed and non empty. We want to show that $A$ is open. Let $\rho_0$ be in $A$ and let us prove that

$$\overline{\Omega^+(u)} \subset \{x \in \overline{\Omega} \mid w_{\rho_0}(x) > 0\} \tag{2.2}$$

Then, using (iv) and the maximum principle one readily gets the desired conclusion. In order to prove (2.2), it suffices to show that $\partial\Omega^+(u) \cap \partial\Omega^+(w_{\rho_0}) = \varnothing$. Let $x_0 \in \partial\Omega$ be in such intersection. Then, there exists a ball $B_\epsilon \in \{w_{\rho_0} = 0\}^\circ$ with $x_0 \in \partial B_\epsilon$. If $u$ vanishes in a boundary neighborhood of $x_0$, then the strict supersolution property and condition (C) imply that

$$u(x) = \alpha(x - x_0, \nu)^+ + o(|x - x_0|), \text{ as } x \to x_0,$$

$$w_{\rho_0}(x) = \beta(x - x_0, \nu)^+ + o(|x - x_0|), \text{ as } x \to x_0,$$

with $\alpha \geq 1 > \beta$. This contradicts $u \leq w_{\rho_0}$. One can argue similarly when $x_0 \in F(u) \cap F(w_{\rho_0})$, using the fact that $u$ is a weak solution. Finally, using (ii) one concludes that (2.2) holds.

□

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$, and let $J$ be the energy functional from Section 1:

$$J(u, \Omega) = \int_\Omega (|\nabla u|^2 + \chi_{\{u>0\}}).$$

Here are some properties of minimizers of $J$.

**Proposition 2.6.** *Let* $u$ *be a minimizer of* $J$ *over* $K = \{v \in H^1(\Omega) | v = w \text{ on } \partial\Omega\}$, *with* $w$ *a non-negative function in* $C^{1,1}(\overline{\Omega})$. *Then the following properties hold:*

(a) $u$ *is a viscosity solution to* (2.1) *in* $\Omega$,
(b) $u$ *is subharmonic in* $\Omega$,
(c) $u \in C(\overline{\Omega})$,
(d) $u$ *satisfies condition* (C).

*Proof.* The proof of the interior statements (a)-(b) can be found in [5], [1] respectively.

Let us prove (c). Denote by $v$ be the harmonic replacement of $w$ in $\Omega$, i.e.

$$\begin{cases} \Delta v = 0 & \text{in } \Omega, \\ v = w & \text{on } \partial\Omega. \end{cases}$$

Then, the weak maximum principle implies $0 \leq u \leq v$ on $\Omega$. Let $\overline{x} \in \partial\Omega^+(u) \cap \partial\Omega$. If $w(\overline{x}) = 0$, since $v$ is continuous on $\overline{\Omega}$, then $v(\overline{x}) = 0$, and from $0 \leq u \leq v$, we then get the continuity of $u$ at $\overline{x}$. If $w(\overline{x}) > 0$, consider a domain $Q_r \subset \Omega$ such that $B_{r/10}(\overline{x})) \cap \partial\Omega \subset \partial Q_r$ and the rescaled set $(1/r)Q_r$ is $C^2$ diffeomorphic to the unit ball uniformly as $r \to 0$. It is a routine matter to construct such a region for sufficiently small $r$, since $\partial\Omega$ is smooth. Let $v_r$ denote the harmonic function in $Q_r$

with $v_r = u$ on $\partial Q_r$. Lemma 3.2 of [1] says[1] that if

$$\int_{\partial Q_r} u d\sigma > Cr^n$$

for sufficiently large $C$ depending on dimension and the $C^2$ norm bounds, then $u > 0$ in $Q_r$. For $r$ sufficiently small,

$$\int_{\partial Q_r} u d\sigma \geq \int_{\partial Q_r \cap \partial\Omega} w d\sigma \geq cw(\overline{x}) r^{n-1} >> r^n$$

Thus $u > 0$ in $Q_r$ for small $r$, and $u$ minimizes the Dirichlet integral in $Q_r$. Consequently, $u = v_r$ in $Q_r$. In particular, $u$ is continuous at $\overline{x}$. (Actually, the proof of Lemma 3.2 of [1] is carried out by showing that $u = v_r$.)

Now, we prove (d). Let $x_0$ be as in condition (C) and let $B_{2R}$ be a ball centered at $x_0$ such that $u \equiv 0$ on $\partial\Omega \cap B_{2R}$. Let us extend $u$ to zero on $B_{2R} \setminus \Omega$. Then $u$ is subharmonic in $B_{2R}$.

We start by proving that $u$ is Lipschitz continuous in $B_R$. For $x \in B_R \cap \Omega^+(u)$, let $d = \mathrm{dist}(x, \partial\Omega^+(u))$ be small. If $B_{2d} \subset \Omega$ then it is known that $u(x) \leq Cd$ (see [1]; Corollary 3.6). In the case when $B_{2d} \cap \partial\Omega \neq \emptyset$, the harmonic majorant $v$ from the beginning of the proof of (c) is Lipschitz continuous and vanishes on $B_{2R} \cap \partial\Omega$. Thus $u \leq v \leq Cd$ on $B_{2d} \cap \Omega$. Hence in all cases $\max_{B_{\frac{d}{2}}} u(x) \leq Cd$. Furthermore, $u$ is harmonic in $B_{\frac{d}{2}}$. This implies that $|\nabla u| \leq C$ on $B_{\frac{d}{4}}$.

Lipschitz continuity and subharmonicity of $u$ in $B_R$, together with the exterior ball condition at $x_0$, imply that:

$$u(x) = \alpha(x - x_0, \nu)^+ + o(|x - x_0|), \text{ as } x \to x_0, \tag{2.3}$$

with $\nu$ inward unit normal at $\partial\Omega$ ([5]; for more details see [9]).

As noted in Remark 3.5 of [1], since $u$ vanishes on $B_{2R} \cap \partial\Omega$, $u$ satisfies the following property. There exists $\varepsilon > 0$, such that, for all $r \leq R/2$,

$$\text{if} \quad \max_{B_{2r}(x_0)\cap\Omega} u \leq \varepsilon r, \quad \text{then} \quad u \equiv 0 \quad \text{on} \quad B_{r/2}(x_0) \cap \Omega.$$

It follows that $\alpha$ in (2.3) is greater than zero.

Finally, $u_0(x) \equiv \alpha(x, \nu)^+$ is the uniform limit (on compact subsets of $\mathbb{R}^n$) of $\frac{1}{r} u(x_0 + rx)$ as $r \to 0$. By [1] this blow-up limit is a minimizer of $J$ on the unit ball $B_1(0)$, among all competitors $v$, such that $v \equiv 0$ on $B_1(0) \cap \{(x, \nu) \leq 0\}$.

Take interior perturbations $u_\epsilon(x) = u_0(x + \epsilon g \nu)$, for $\epsilon > 0$, $g \in C_0^\infty(B_1), g \leq 0$. Because $\alpha > 0$, the first variation of $J$ is ([1]; Theorem 2.5)

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon}(J(u_\epsilon) - J(u_0)) = (1 - \alpha^2) \int_{(x,\nu)=0} g d\sigma.$$

Since the first variation is non-negative for $g \leq 0$, we conclude that $\alpha \geq 1$.

□

[1] Lemma 3.2 of [1] is stated for balls $B_r$, but applies to any region which, when rescaled by the factor $1/r$, is $C^{1,\alpha}$ diffeomorphic to the unit ball for some $\alpha > 0$. What is used in the proof is that the rescaled region is uniformly biLipschitz to $B_1$ and one other fact that is a well-known consequence of $C^{1,\alpha}$ smoothness, namely a linear lower bound on the decay rate of the nonnegative harmonic function $v_r$:

$$v_r(x) \geq c \frac{\mathrm{dist}(x, \partial Q_r)}{r} r^{-(n-1)} \int_{\partial Q_r} v_r d\sigma$$

with a constant $c$ that is uniform as $r \to 0$.

From now on, denote by $U$ the critical point of $J$ introduced in Section 1. Let $B$ be a ball on $\mathbb{R}^n$, centered at the origin. In Sections 3 and 4, respectively, we prove the following results, which together give Theorem 1.1.

**Theorem 2.7.** *Let $u$ minimize $J(\cdot, B)$ over $K = \{v \in H^1(B) | v = U \text{ on } \partial B\}$. In dimensions $7 \leq n \leq 20$, we have $u \geq U$.*

**Theorem 2.8.** *Let $u$ minimize $J(\cdot, B)$ over $K = \{v \in H^1(B) | v = U \text{ on } \partial B\}$. In dimensions $n = 7, 9, 11, 13, 15$, we have $u \leq U$.*

The proofs of Theorem 2.7 and Theorem 2.8 are based on the comparison of $u$ with appropriate families of subsolutions and supersolutions, via the deformation lemmas 2.3 and 2.5. By Proposition 2.6, a minimizer $u$ of $J(\cdot, B)$ is a weak solution to (2.1) in $B$. The main issue consists of constructing a strict subsolution $V$, and a strict supersolution $W$ to (2.1) on $\mathbb{R}^n$, which satisfy $V \leq U \leq W$. By dilation of $V$ and $W$, we then produce the desired families of subsolutions and supersolutions.

The method indicates that Theorem 2.7 and Theorem 2.8 hold for all $n \geq 7$. We have only stated them in the dimensions in which we have confirmed them numerically. In particular the proof of Theorem 2.8, which is the harder of the two, becomes easier as $n \to \infty$ (see Section 5).

## 3. Construction of a subsolution

We will construct a subsolution $V$ to (2.1) in $\mathbb{R}^n$ by perturbing the function $Z$ defined in Section 1. $Z$ is expressed in terms of a Legendre function. Let $f_n(t)$ be a nonzero even function of $t$, satisfying the Legendre equation

$$(1-t^2)f_n''(t) + (1-n)tf_n'(t) + (n-1)f_n(t) = 0, \quad -1 < t < 1, \tag{3.1}$$

($f_n$ is unique up to scalar multiple.) Let $t_n$ be its smallest positive zero, and assume that $f_n$ is positive on the open interval $(-t_n, t_n)$. Define

$$Z(x) = |x| f_n\left(\frac{x_n}{|x|}\right)$$

To determine the normalizing constant, we need to compute $|\nabla Z|^2$ on $\partial\Gamma \setminus \{0\}$. We have:

$$\partial_{x_i} Z(x) = \frac{x_i}{|x|} f_n\left(\frac{x_n}{|x|}\right) + |x| f_n'\left(\frac{x_n}{|x|}\right) \cdot \left(\frac{\delta_{i,n}|x| - x_n x_i |x|^{-1}}{|x|^2}\right), i = 1, ...n.$$

where $\delta_{i,j}$ is the Kronecker symbol. Therefore,

$$|\nabla Z|^2 \equiv (1-t_n^2)(f_n'(t_n))^2 \quad \text{on } \partial\Gamma \setminus \{0\}.$$

Set $c_n = (1-t_n^2)(f_n'(t_n))^2$, then $|c\nabla Z|^2 \equiv 1$ on $\partial\Gamma \setminus \{0\}$ for $c = 1/\sqrt{c_n}$. We have not yet specified a choice of $f_n$, but we will eventually do this in a conventional way for Legendre functions. When we do so, $c_n$ will be a specific dimensional constant.

Now define

$$V(x) = \frac{1}{\sqrt{c_n}}\left(Z(x) - |x|^{\alpha_n} g_{\alpha_n}\left(\frac{x_n}{|x|}\right)\right) \tag{3.2}$$

where $\alpha_n < 1$ is a parameter to be chosen later, and $g_\alpha$ is a positive and even function of t, satisfying the Legendre equation

$$(1-t^2)g_\alpha''(t) + (1-n)tg_\alpha'(t) + \alpha(\alpha+n-2)g_\alpha(t) = 0, \quad \text{on } (-1,1), \tag{3.3}$$

($g_\alpha$ is unique up to scalar multiple.) Thus, the function $Y(x) = |x|^{\alpha_n} g_{\alpha_n}\left(\frac{x_n}{|x|}\right)$ is a positive harmonic function in the cone $\Gamma$ such that $Y$ is a constant multiple of $|x|^{\alpha_n}$ on $\partial\Gamma$ and $Y$ is homogeneous of degree $\alpha_n$. The function $V$ is then harmonic in the cone $\Gamma$. We want to show that $V_\nu \geq 1$ on $\partial\{V > 0\}$. Then, we can conclude that $V^+$ is a weak subsolution to problem (2.1) in $\mathbb{R}^n$.

Toward this aim, let us compute $|\nabla V|^2$. For simplicity, denote by $\alpha = \alpha_n$. For $i = 1, ..., n$,

$$\sqrt{c_n}\partial_{x_i} V(x) = \partial_{x_i} Z - \alpha|x|^{\alpha-2} x_i g_\alpha\left(\frac{x_n}{|x|}\right) - |x|^\alpha g_\alpha^{'}\left(\frac{x_n}{|x|}\right)\cdot\left(\frac{\delta_{i,n}|x| - x_n x_i |x|^{-1}}{|x|^2}\right).$$

Hence,

$$c_n|\nabla V|^2(x) = \left[f_n\left(\frac{x_n}{|x|}\right) - \alpha|x|^{\alpha-1} g_\alpha\left(\frac{x_n}{|x|}\right)\right]^2 + \left(1 - \frac{x_n^2}{|x|^2}\right)\left[f_n^{'}\left(\frac{x_n}{|x|}\right) - |x|^{\alpha-1} g_\alpha^{'}\left(\frac{x_n}{|x|}\right)\right]^2,$$

and consequently, for $x$ on $\partial\{V > 0\}$

$$|\nabla V|^2(x) = \frac{1}{c_n} G_n\left(\alpha, \frac{x_n}{|x|}\right), \tag{3.4}$$

where

$$G_n(\alpha, t) = (1-\alpha)^2 f_n^2(t) + (1 - t^2)\left(f_n^{'}(t) - \frac{f_n(t)}{g_\alpha(t)} g_\alpha^{'}(t)\right)^2. \tag{3.5}$$

Whenever this does not create confusion, we will write $G_n(t)$ for $G_n(\alpha, t)$.

In order to conclude that $V^+$ is a weak subsolution, we have to verify that $|\nabla V|^2 \geq 1$ on $\partial\{V > 0\}$. By definition, $f_n(t_n) = 0$, so $G_n(\alpha, t_n) = c_n$. Therefore by (3.4), the statement $|\nabla V|^2 \geq 1$ on $\partial\{V > 0\}$ is equivalent to requiring that $G_n$ achieves its absolute minimum on $[-t_n, t_n]$ at the boundary points.

Recall that $f_n$, $g_\alpha$ are even functions of $t$, hence $G_n$ is also an even function of $t$. Therefore, 0 is either a local minimum or a local maximum. In particular, computing $G_n^{''}(0)$, using the properties of $f_n$ and $g_\alpha$, one gets:

$$\begin{cases} (\alpha + n - 1)^2 > n - 1 & 0 \text{ is a minimum point,} \\ (\alpha + n - 1)^2 < n - 1 & 0 \text{ is a maximum point.} \end{cases} \tag{3.6}$$

We study the behavior of the function $G_n$, for different values of $n$, and various choices of the parameter $\alpha$ in the ranges above. It turns out that $G_n$ has two other interior critical points. According to (3.6), depending on the choice of $\alpha$,the critical point of $G_n$ at $t = 0$ can be either a relative maximum or minimum. Our goal is that the end points be the absolute minimum. For that reason, one might think that it is wise to make 0 a relative maximum. But in fact, if 0 is a relative maximum then $G_n$ has its absolute minimum at interior points in the interval:
Instead, we will choose $\alpha$ for which $G_n$ has a relative minimum at 0, and $G_n(\alpha, 0) \geq G_n(\alpha, t_n)$. This is the first place where the dimensional restriction $n \geq 7$ appears. For $7 \leq n \leq 20$, numerical computation shows that we can find a value $\alpha_n$ for which $G_n(\alpha_n, 0) = G_n(t_n)$, and $G_n$ attains its absolute minimum at three points $t = \pm t_n$ and $t = 0$ (see Figure 2.)

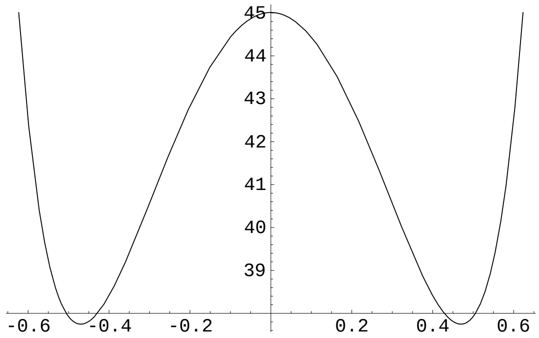


FIGURE 1. Graph of $G_5(\alpha_5, t)$ on $[-t_5, t_5]$, $\alpha_5 < -2$.

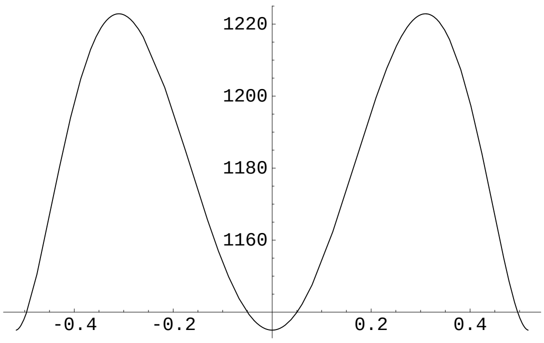


FIGURE 2. Graph of $G_7(\alpha_7, t)$ on $[-t_7, t_7]$, $\alpha_7 \simeq -3.211$.

For the numerical computations involved in analysis described above, we have used Mathematica. To prevent round-off errors and to ensure that we obtain a strict subsolution we will ultimately choose $\widetilde{\alpha_n}$ slightly smaller than $\alpha_n$.

Let us describe the details for the case $n$ odd and for $\alpha$ such that $G_n(\alpha, 0) = G_n(t_n)$. We will use the notations from [7]. Set $\nu = (n-1)/2$, and $\mu = (n-3)/2$. We have :

$$f_n(t) = (1-t^2)^{-\frac{n-3}{4}} Q^\mu_\nu(t),$$

and

$$\begin{array}{ccc} t_3 = 0.833557 & t_5 = 0.623175 & t_7 = 0.517331 \\ t_9 = 0.451615 & t_{11} = 0.405841 & t_{13} = 0.371631 \\ t_{15} = 0.34482 & t_{17} = 0.323078 & t_{19} = 0.304987 \end{array}$$

Although Mathematica can compute derivatives symbolically and numerically, it is more efficient and reliable to use formulas for derivatives computed by hand. Formulas for the derivatives of Legendre functions can be found in [7]. For convenience of the reader we report the two formulas which we have used in this context. Here we denote by $L$ either one of the Legendre functions $P$ and $Q$.

$$(3.7)\quad \begin{cases} \frac{d}{dt}L_\nu^\mu(t) = -\mu t(1-t^2)^{-1}L_\nu^\mu(t) - (1-t^2)^{-\frac{1}{2}}L_\nu^{\mu+1} \\ \\ \frac{d}{dt}L_\nu^\mu(t) = \mu t(1-t^2)^{-1}L_\nu^\mu(t) + (\nu+\mu)(\nu-\mu+1)(1-t^2)^{-\frac{1}{2}}L_\nu^{\mu-1}(t). \end{cases}$$

Thus, $\alpha = \alpha_n$ is given by

$$\alpha = \alpha_n = 1 - (1-t_n^2)^{-\frac{n-3}{4}} Q_\nu^\nu(t_n)/Q_\nu^\mu(0).$$

Some numerical values of $\alpha$ are reported below.

$$\begin{array}{ccc} \alpha_3 = -1.71506 & \alpha_5 = -2.35453 & \alpha_7 = -3.21122 \\ \alpha_9 = -3.91985 & \alpha_{11} = -4.5382 & \alpha_{13} = -5.09399 \\ \alpha_{15} = -5.60307 & \alpha_{17} = -6.07558 & \alpha_{19} = -6.51842 \end{array}$$

The function $g_\alpha$ has the following representation:

$$g_\alpha(t) = \frac{1}{2}(1-t^2)^{-\frac{n-3}{4}}\left(P_{\alpha+\mu}^{-\mu}(t) + P_{\alpha+\mu}^{-\mu}(-t)\right).$$

Combining the formulas above,

$$G_n(t) = (1-t^2)^{-\frac{n-3}{2}}\{[Q_\nu^\nu(t) + \alpha(\alpha+n-2)\cdot Q_\nu^\mu(t)Z_n(t)E_n(t)]^2 + (1-\alpha)^2(Q_\nu^\mu(t))^2\}$$

where

$$Z_n(t) = P_{\alpha+\mu}^{-\nu}(t) - P_{\alpha+\mu}^{-\nu}(-t),$$

and

$$E_n(t) = (P_{\alpha+\mu}^{-\mu}(t) + P_{\alpha+\mu}^{-\mu}(-t))^{-1}.$$

Plotting the graph of the function $G_n$ for $n = 7$, we observe that $G_n$ attains its absolute interior minimum at $t = 0$. The graph of $G_7(\alpha_7, t)$ is plotted in Figure 2. Here is the graph of $G_7(\widetilde{\alpha_7}, t)$ :

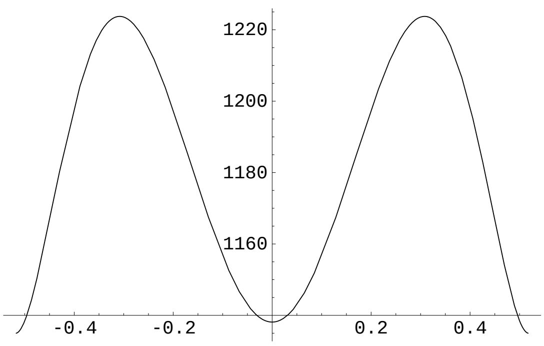


FIGURE 3. Graph of $G_7(\widetilde{\alpha_7}, t)$ on $[-t_7, t_7]$, $\widetilde{\alpha_7} \simeq -3.215$.

By evaluation $G_7(\widetilde{\alpha_7}, 0)$ is larger than $G_7(\widetilde{\alpha_7}, \pm t_7)$. The function $G_7(\widetilde{\alpha_7}, t)$ is defined on $(-1, 1)$. A close-up in a small neighborhood of $\pm t_7$ shows that the graph is strictly above $G_7(\widetilde{\alpha_7}, t_7)$ in the open interval $(-t_7, t_7)$. An additional check that

$G_7(\widetilde{\alpha_7}, t)$ is strictly monotone around $\pm t_7$ is performed by plotting the derivative $\partial_t G_7(\widetilde{\alpha_7}, t)$.

We take the same approach whenever a certain upper/lower bound is shown to hold by plotting the graph of a function on an interval. If the function is known to achieve its maximum/minimum value at some point in the interval, we perform a close-up of the graph around the correspondent maximum/minimum point, and study the behavior of the first derivative there.

*Remark* 3.1. The calculations above have been carried out for $7 \leq n \leq 20$, which also shows the stability of our method. In the case $n$ even, we proceed as in the case $n$ odd, but using the Legendre function $P$ for the representation of $f_n$. The graph of $G_8(\widetilde{\alpha_8}, t)$ is displayed in Figure 4.

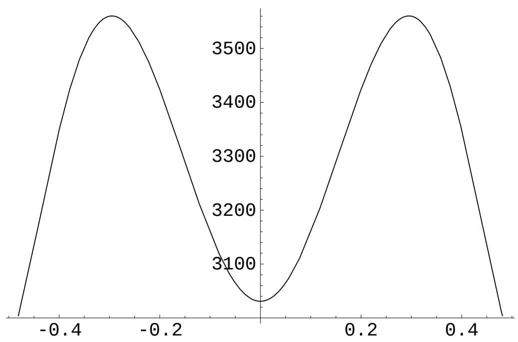


FIGURE 4. Graph of $G_8(\widetilde{\alpha_8}, t)$ on $[-t_8, t_8]$, $\widetilde{\alpha_8} \simeq -3.6$.

*Proof of Theorem* 2.7. For $\alpha$ slightly smaller than $\alpha_n$, $7 \leq n \leq 20$, set

$$V_\rho(x) = \frac{1}{\rho} V^+(\rho x) = \frac{1}{\sqrt{c_n}} \left( Z(x) - \rho^{\alpha-1} Y(x) \right)^+ .$$

$V_\rho$ is a continuous (in $\rho$) family of subsolutions to (2.1) on $B$. Furthermore, because $\alpha$ is negative, $\inf_B Y$ is positive. Therefore one can choose an $a > 0$ small enough so that $V_a^+ \equiv 0$ in $B$. Let $b > a$ and apply Lemma 2.3 to $u$ and $V_\rho$, on $[a, b]$. Then, $u \geq V_\rho$ for all $\rho > 0$. Moreover $V_\rho$ converges to $U$ on $B$, as $\rho \to +\infty$. Hence $u \geq U$ on $B$.

$\square$

## 4. Construction of a supersolution

We will construct a weak, strict supersolution $W$ to (2.1) in $\mathbb{R}^n$, whose positive phase contains the set $\Gamma$. We start by performing a change of variables.

Let $W(x) = w(x_n, |x'|)$, where $x' = (x_1, ..., x_{n-1})$ and $w(s, r)$ is an even function of $s$. Then, $W$ is a weak strict supersolution to (2.1) in $\mathbb{R}^n$, if and only if $w$ solves

$$\begin{cases} \mathcal{L}w = \dfrac{\partial^2}{\partial s^2} w + \dfrac{\partial^2}{\partial r^2} w + \dfrac{(n-2)}{r} \dfrac{\partial}{\partial r} w \leq 0 & \text{on } \{w > 0\}, \\ |\nabla w|^2 < 1 & \text{on } \partial\{w > 0\}. \end{cases} \tag{4.1}$$

In this coordinate system, the function $Z$ in the previous sections, is given by $Z(x) = z(x_n, |x'|)$, where

$$z(s,r) = \sqrt{s^2+r^2} f_n\left(\frac{s}{\sqrt{s^2+r^2}}\right),$$

while $\Gamma$ is described by

$$\Gamma = \{(s,r) \in \mathbb{R}\times\mathbb{R}^+ : d_n|s| < r\}, \quad d_n = \sqrt{1-t_n^2}/t_n.$$

Recall the $f_n$ is defined in (3.1), and $t_n$ is its smallest positive zero.

We construct $W$ in two pieces $w_1$ and $w_2$ above and below the line $r = \overline{r}$, with $\overline{r}$ chosen later (see Figure 5).

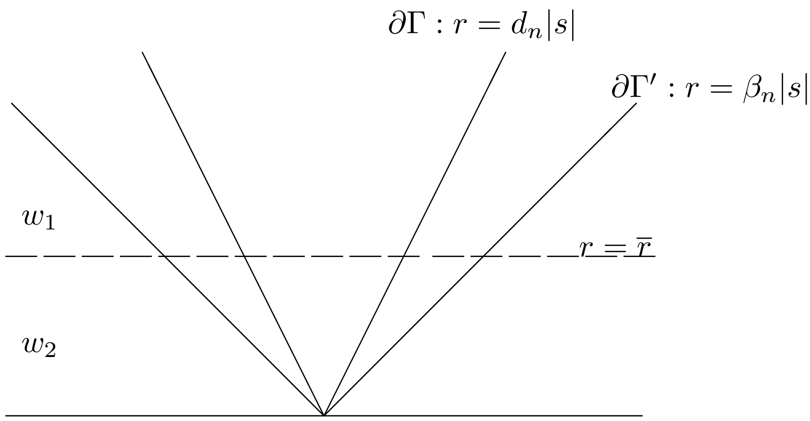


FIGURE 5.

*Step 1.* For $0 < \beta_n < d_n$, consider the cone $\Gamma' = \{(s,r) \in \mathbb{R}\times\mathbb{R}^+ : \beta_n|s| < r\}$ and set $\gamma_n = 1/\sqrt{1+\beta_n^2}$. $\beta_n$ will be chosen later, so that $f_n(\gamma_n) < 0$. Set

$$k(s,r) = (s^2+r^2)^{\frac{\tau_n}{2}} g_{\tau_n}\left(\frac{s}{\sqrt{s^2+r^2}}\right).$$

We want $k$ to be $\mathcal{L}$-harmonic and positive in the cone $\Gamma'$. Thus $g_{\tau_n}$ solves the same equation as $g_\alpha$ in (3.3). Moreover we require $\tau_n < 1$, and $g_{\tau_n}$ to be even and positive on $[-\gamma_n, \gamma_n]$. For simplicity, denote $\tau = \tau_n$.

Now, set

$$w_1(s,r) = \frac{1}{\sqrt{c_n}}\{z(s,r) + k(s,r)\}$$

with $c_n$ the normalizing constant from Section 3. Then equations (3.1) and (3.3) imply that $\mathcal{L}w_1 = 0$ in $\Gamma'$. We aim to choose $\tau$ and $\gamma_n$ so that $w_1$ is a weak strict supersolution to (4.1) away from the singular axis $r = 0$, which can then be extended to a supersolution in the whole upper plane.

The level set $\{w_1 = 0\}$ intersects $\partial\Gamma'$ at the points $(\pm\overline{s}, \overline{r})$, where

$$\overline{s} = \gamma_n\left(-\frac{g_\tau(\gamma_n)}{f_n(\gamma_n)}\right)^{\frac{1}{1-\tau}}, \quad \overline{r} = \beta_n\overline{s}. \tag{4.2}$$

In order to guarantee that $\{w_1(s,r)=0\}$ is inside $\Gamma'$ for $r>\overline{r}$, we need to know that the slope $A(\beta_n)$ of the level set $w_1(s,r)=0$ at $(\overline{s},\overline{r})$ is greater than $\beta_n$ (the slope of $\partial\Gamma'$). Therefore we will choose $\gamma_n$ so that $A(\beta_n)>\beta_n$.

Denote by $\Omega=\Gamma'\cap\{r>\overline{r}\}$. The same computations as in Section 3 show that on $\partial\{w_1>0\}\cap\overline{\Omega}$

$$c_n|\nabla w_1|^2(s,r)=G_n(\tau,s/\sqrt{s^2+r^2})$$

where we recall from (3.5) that

$$G_n(\tau,t)=(1-\tau)^2 f_n(t)^2+(1-t^2)\left[f_n'(t)-\frac{f_n(t)g_\tau'(t)}{g_\tau(t)}\right]^2.$$

The strict supersolution free boundary condition is satisfied if $G_n(t)<c_n$ on $(t_n,\gamma_n]$. Since $c_n=G_n(t_n)$, we are requiring $G_n$ to decrease in a right neighborhood of $t_n$. As before, we analyze the behavior of $G_n$ using Mathematica. The value of $\alpha=\alpha_n$ in the perturbation of $U$ by $g_\alpha$ used to construct a subsolution $V$ does not work for the supersolution $W$. Although $w_1$ can be defined as a supersolution for $r>\overline{r}$ for a wide range of $\alpha$'s including $\alpha_n$, we need another value $\alpha=\tau_n$ in order to link $w_1$ to a supersolution $w_2$ defined for $0\le r\le\overline{r}$. Thus we need to describe $w_2$ before fixing $\tau_n$.

*Step 2.* Let us set $h(s)=w_1(s,\overline{r})$, and define:

$$w_2(s,r)=y(r)h\left(\frac{s}{v(r)}\right),$$

where the functions $y(r)$ and $v(r)$ will be chosen positive and even on the real line, and satisfying end point conditions

$$y(\overline{r})=v(\overline{r})=1 \tag{4.3}$$

and

$$v'(\overline{r})=\frac{1}{\overline{s}A(\beta_n)}. \tag{4.4}$$

The latter condition guarantees that the level curves $w_1(s,r)=0$ and $w_2(s,r)=0$ have the same slope at the points $(\pm\overline{s},\overline{r})$. Define

$$w(s,r)=\begin{cases} w_1(s,r) & \text{in } \overline{\Omega},\\ w_2(s,r) & \text{in } \{r\le\overline{r}, r\le\overline{s}v(r)\}.\end{cases}$$

We will prove that we can find $y(r),v(r)$ so that $w^+(s,r)$, extended to zero outside its positive phase, is the desired weak strict supersolution to (4.1) and, by construction $\Gamma\subset\{w>0\}$. Toward this aim we need to verify that:

$$\mathcal{L}w_1\le 0 \quad \text{in } \Omega^+(w_1) \tag{4.5}$$

$$|\nabla w_1|^2<1 \quad \text{on } \partial\{w_1>0\}\cap\overline{\Omega} \tag{4.6}$$

$$\mathcal{L}w_2\le 0 \quad \text{in } \{|r|<\overline{r},|s|<\overline{s}v(r)\}, \tag{4.7}$$

$$|\nabla w_2|^2<1 \quad \text{on } \{|s|=\overline{s}v(r),|r|\le\overline{r}\}, \tag{4.8}$$

$$\frac{\partial}{\partial r}w_1(s,r)|_{\overline{r}}<\frac{\partial}{\partial r}w_2(s,r)|_{\overline{r}} \quad \text{on } (-\overline{s},\overline{s}). \tag{4.9}$$

We remark that condition (4.9) is needed to guarantee that the piecewise function $w$ is the minimum of $w_1$ and $w_2$ in $\{w>0\}$, across $\{r=\overline{r}\}$. Thus $w$ is a supersolution in $\{w>0\}$.

Condition (4.5) follows already from Step 1, while condition (4.6) will follow from an appropriate choice of the parameters $\tau_n$ and $\gamma_n$. The free boundary condition (4.8) will be obtained as a consequence of condition (4.6). Indeed, we choose $y(r)$ so that $|\nabla w_2|^2|_{s=\overline{s}v(r)} = |\nabla w_1|^2|_{(\overline{s},\overline{r})}$, for all $r \leq \overline{r}$. In other words $|\nabla w_2|$ is constant along the free boundary $|s| = \overline{s}v(r)$ which specifies $y(r)$ as follows

$$y(r) = \sqrt{\left(1 + \frac{1}{A(\beta_n)^2}\right)} \frac{v(r)}{\sqrt{\overline{s}^2 v'(r)^2 + 1}}. \tag{4.10}$$

Condition (4.7) will be fulfilled with a correct choice of $v(r)$. To confirm this we need a formula for $\mathcal{L}w_2$:

$$\begin{aligned}\mathcal{L}w_2(s,r) = \frac{y(r)}{v(r)^2} h''\left(\frac{s}{v(r)}\right)\left\{1 + s^2 \frac{v'(r)^2}{v(r)^2}\right\} \\ - \frac{sy(r)}{v(r)^2} h'\left(\frac{s}{v(r)}\right) M(r) + \frac{y(r)}{v(r)} h\left(\frac{s}{v(r)}\right) N(r)\end{aligned}$$

where

$$M(r) = v(r)\left\{2\frac{y'(r)}{y(r)}\frac{v'(r)}{v(r)} + \frac{v''(r)}{v(r)} - 2\frac{v'(r)^2}{v(r)^2} + \frac{n-2}{r}\frac{v'(r)}{v(r)}\right\}$$

and

$$N(r) = v(r)\left\{\frac{y''(r)}{y(r)} + \frac{n-2}{r}\frac{y'(r)}{y(r)}\right\}.$$

In order to determine $v(r)$ so that $\mathcal{L}w_2 \leq 0$, we first study the behavior of the first and second derivative of $h$. Recall that $h(s) = w_1(s, \overline{r})$, and the formula for $w_1$ involves the functions $f_n$ and $g_\tau$ for which we can compute explicit representation formulas in Mathematica, when varying $n$ and $\tau$.

We describe the case when $n = 7$. We observe that, for various choices of the parameters $\tau$ and $\gamma_7$, which are compatible with the free boundary condition (4.6), the functions $\xi h'(\xi)$ and $h''(\xi)$ are both non positive on $[-\overline{s}, \overline{s}]$. In particular, for

$$\tau = -1.756, \quad \gamma = 0.62,$$

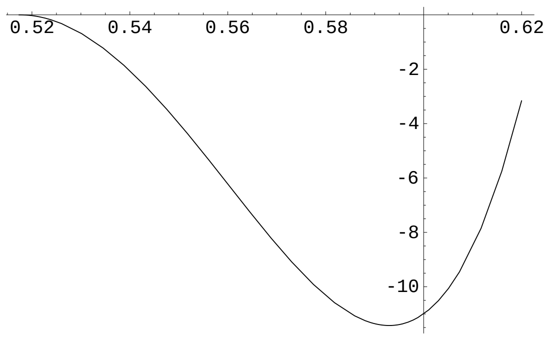


FIGURE 6. Graph of $G_7(\tau,t) - G_7(\tau,t_7)$ on $[t_7, \gamma_7]$.

Therefore, if we assume that $v(r) \leq 1$, and that $M(r)$ and $N(r)$ are both positive, then $\mathcal{L}w_2$ is majorized by

$$K(s,r) = \frac{y(r)}{v(r)^2}\{h''(s/v(r)) - s/v(r)h'(s/v(r))\overline{M} + h(s/v(r))\overline{N}\}$$

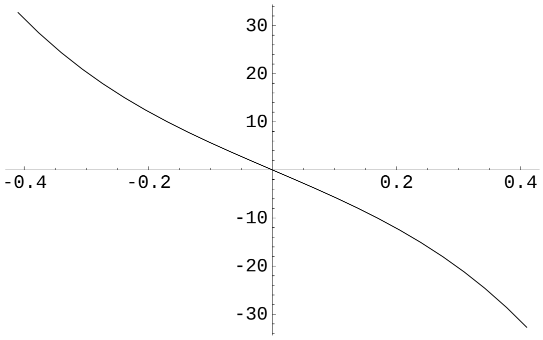


FIGURE 7. Graph of $h'(\xi)$ on $[-\overline{s}, \overline{s}]$.

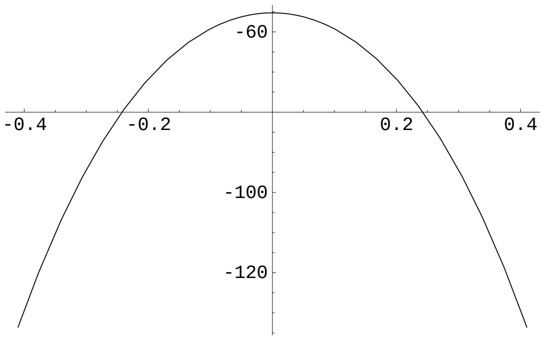


FIGURE 8. Graph of $h''(\xi)$ on $[-\overline{s}, \overline{s}]$.

with $\overline{M}$ and $\overline{N}$ the maximum values of $M$ and $N$ respectively. Let $v(r)$ be an even polynomial. We aim to determine $v(r)$ so that,

$$\begin{cases} v(r) \leq 1 & \text{on } [-\overline{r}, \overline{r}] \\ M(r), N(r) \geq 0 & \text{on } [-\overline{r}, \overline{r}] \\ \overline{K}(\xi) = h''(\xi) - \xi h'(\xi)\overline{M} + h(\xi)\overline{N}) \leq 0 & \text{on } [-\overline{s}, \overline{s}]. \end{cases} \tag{4.11}$$

Since $v(r)$ must satisfy conditions (4.3) and (4.4), we will assume that $v(r)$ is a fourth degree polynomial so as to have a one extra parameter of freedom. Precisely, set

$$v(r) = a_7 r^4 + b_7 r^2 + v_7.$$

We need to determine $a_7$ so that all (4.11) is satisfied, when $\tau$ and $\gamma_7$ are chosen to obtain the positivity of the functions $\xi h'(\xi)$ and $h''(\xi)$. In particular, since $M$ and $N$ are both even, we will look for $a_7$ such that $\overline{M} = M(0)$ and $\overline{N} = N(0)$. Moreover, we also need (4.9) to hold. As in Section 3 (see comment before (3.7)) we compute explicit formulas for all the derivatives involved, rather than relying on Mathematica . All the required conditions are non linear inequalities which must be satisfied by $a_7$. Our purpose is to choose parameters $\tau_7$ and $\gamma_7$, compatible with the free boundary condition (4.6), and so that such an $a_7$ exists. We report the specific numerical values, for which the method described above succeeds. The formulas for $f_7$ and $g_{\tau_7}$ are the same as in the previous section. We have,

$$t_7 = 0.517331 \qquad \tau_7 = -1.756 \qquad \gamma_7 = 0.62 \qquad a_7 = -0.3664026$$

which yield

$$\overline{s} = 0.410006 \qquad \overline{r} = 0.518857 \qquad b_7 = 0.757322 \qquad v_7 = 0.822675.$$

Figures 6,7 and 8 use these values. Figure 6 is the free boundary condition (4.6), Figure 7 shows the behavior of $\xi h'(\xi)$ and Figure 8 confirms the positivity of $h''(\xi)$. Now, in Figures 9-12, we exhibit the graphs of the functions $v, M, N$ and $\overline{K}$, to show that (4.11) is satisfied. Finally in Figure 13, we display the graph of $\mathcal{W}(s) = \partial_r(w_1(s,r) - w_2(s,r))|_{r=\overline{r}}$, to show that the compatibility condition (4.9) holds.

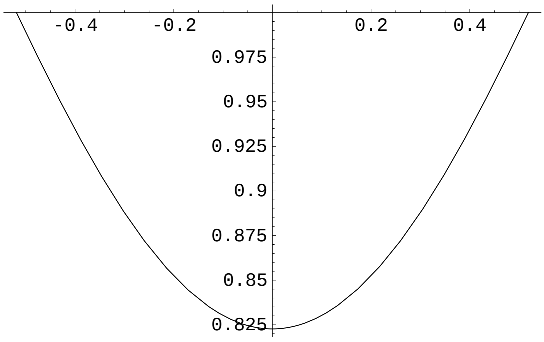


FIGURE 9. Graph of $v(r)$ on $[-\overline{r}, \overline{r}]$.

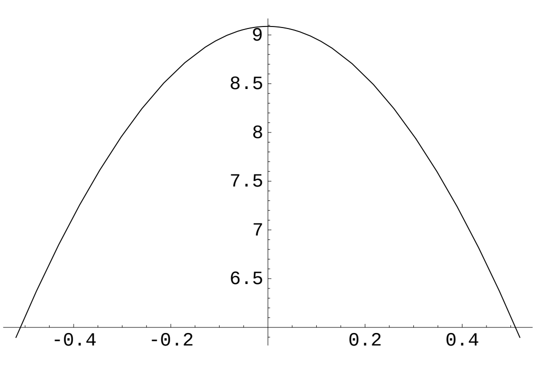


FIGURE 10. Graph of $M(r)$ on $[-\overline{r}, \overline{r}]$.

*Remark* 4.1. In Figure 6, $G(\tau, t) - G(\tau, t_7) < 0$ on $(t_7, \gamma_7]$, hence the free boundary condition is achieved in a strict sense. Also, from Figure 12 we obtain that $\mathcal{L}w_2 < 0$. Both inequalities are needed to be strict, in order to prevent numerical errors. Figure 13 does not display the graph of $\mathcal{W}$ to sufficient accuracy. Figure 14 shows a close-up of a neighborhood of $\overline{s}$.

We remark that $\mathcal{W}(\overline{s}) = -1.688 \times 10^{-14}$, which is strictly negative given that the both the machine precision and the epsilon precision of Mathematica are 16 decimal places. Moreover the value $\mathcal{W}(\overline{s})$ is stable for small perturbations of $\tau_7$

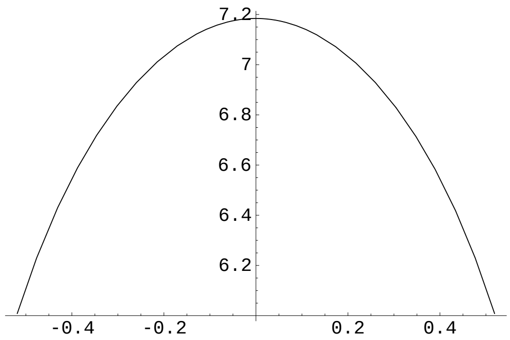


FIGURE 11. Graph of $N(r)$ on $[-\overline{r}, \overline{r}]$.

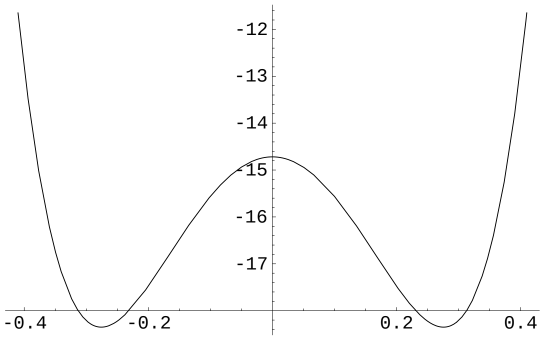


FIGURE 12. Graph of $\overline{K}(\xi)$ on $[-\overline{s}, \overline{s}]$.

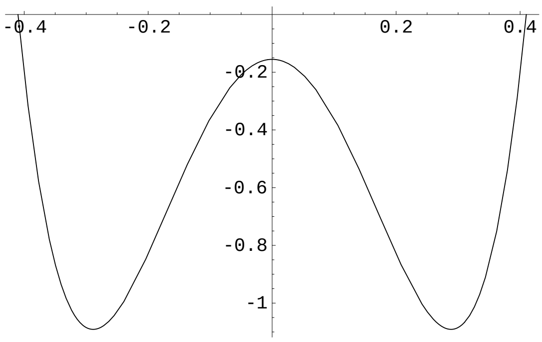


FIGURE 13. Graph of $\mathcal{W}(s)$ on $[-\overline{s}, \overline{s}]$.

and $\gamma_7$. It is natural that the computation should require the most accuracy in the critical dimension $n = 7$. Indeed already when $n \geq 11$ we only need 5 decimal places to confirm the necessary inequalities.

Similar computations have been carried on for $n = 9$. For $n = 11$ and higher, we can simplify our algorithm, as described in the next section.

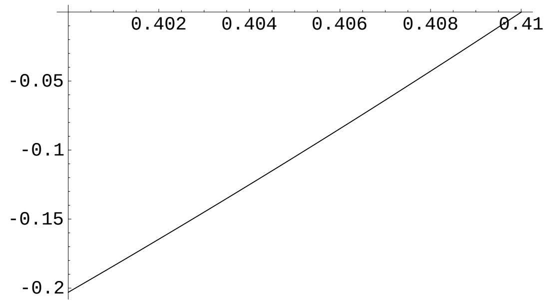


FIGURE 14. Graph of $\mathcal{W}(s)$ on $[-0.4, \bar{s}]$.

*Proof of Theorem* 2.8. Set

$$W_\rho(x) = \frac{1}{\rho} W(\rho x).$$

$W_\rho$ is a continuous (in $\rho$) family of supersolutions to (2.1) on $B$, for any compact interval $[a, b]$, $0 < a < b$. In particular, if $a$ is sufficiently small, then $W_a \geq \sup_{\partial B} U = \sup_B u$. Then, we can apply Lemma 2.5 to $u$ and $W_\rho$, to obtain $u \leq W_\rho$ on $B$, for all $\rho \geq a$. Furthermore, $W_\rho$ converges to $U$ on $B$, as $\rho \to +\infty$. Hence $u \leq U$ on $B$. $\square$

## 5. Remark on Higher dimensional cases

The method described in Section 4 does not provide systematic explicit formulas for $\tau_n$, $\gamma_n$ and $a_n$ in terms of the dimension $n$. We searched a 3-parameter space, where each of the parameters is constrained by a set of nonlinear inequalities, and we found values in the intersection of all these inequalities. On the other hand, in the case when $n \geq 11$, $n$ odd, we have developed a simpler approach. This method reduces the number of parameters to 2, $\tau_n$ and $\gamma_n$. It also simplifies the system of inequalities that such parameters need to satisfy. Finally it provides an explicit formula for $\tau_n$.

We will describe this method briefly, using the same notation as in Section 4. The construction of $W$ is again piecewise. In the first step, in the definition of $w_1$, we replace the function $g_{\tau_n}$, with the function $\phi_n$,

$$\phi_n(r) = (1 - r^2)^{\frac{1-n}{2}}.$$

To confirm the supersolution property $\mathcal{L}w_1 < 0$ in $\Omega^+(w_1)$, for $n \geq 7$ we need

$$\begin{cases} \gamma_n^2 < -\frac{(n-1)+\tau_n(\tau_n+n-2)}{(n-1)-\tau_n(\tau_n+n-2)} = \delta_n^2 & , \\ \\ \frac{2-n-\sqrt{n^2-8n+8}}{2} < \tau_n < \frac{2-n+\sqrt{n^2-8n+8}}{2}. \end{cases} \tag{5.1}$$

Moreover recall that we also need $\gamma_n > t_n$. For $n$ large it is easy to see that there are values of $\gamma_n$ and $\tau_n$ satisfying $\gamma_n > t_n$ and (5.1).

Furthermore, the free boundary condition is easy to check since the form of $\phi_n$ was chosen so that the corresponding function $G_n$ as in (3.5) decreases in a small neighborhood of $t_n$.

We will chose the value $\tau_n = \frac{2-n}{2} + 1$. In the second step, we define $w_2$ as in Section 4, but we choose $v(r)$ to be an even polynomial of degree 2. Therefore $v(r)$ it is completely determined by conditions (4.3) and (4.4). For $n \geq 11$, we can determine $\gamma_n$ close to $t_n$, and so that $M$ and $N$ are positive and $v(r) \leq 1$ on $[-\bar{r}, \bar{r}]$. $\mathcal{L}w_2$ is then majorized by the negative function:

$$\widetilde{K}(r) = \frac{y(r)}{v(r)^2} h''(0) - y(r)\bar{s}h'(\bar{s})M(r) + y(t)h(0)N(r).$$

Here are the numerical values of $\gamma_n$, for $n = 11, 13, 15$:

$$\gamma_{11} = 0.47 \qquad \gamma_{13} = 0.43 \qquad \gamma_{15} = 0.38375.$$

For such choices of $\tau_n, \gamma_n$ the compatibility condition (4.9) is also satisfied. In this case the function $\mathcal{W}$ from Figure 12 is strictly negative on the order than $-10^{-2}$. Hence $W$ is a supersolution to (2.1) on $\mathbb{R}^n$.

We want to note that as $n \to \infty$, $\tau_n \neq \alpha_n$. Roughly speaking $\tau_n > \frac{2-n}{2}$ while $\alpha_n < \frac{2-n}{2}$.

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