COLUMBIA – BARNARD

Mathematics Prize Exam

March 24th, 2011

This is a three-hour exam. It has eight questions. **Each question is worth 10 points**. Submit your paper even if you have done no more than one or two problems. For maximal credit give complete reasons for your answers. However, partial credit will be given for signicant progress made on a problem.

Please fill in the cover page and copy your exam number to each answer form. Do not staple together different problems.

1. Show that there is more than one function $y : \mathbb{R} \to \mathbb{R}$ such that

$$y' = y^{2/3}, \quad y(0) = 1.$$

2. Let f be a real valued C^1 function on $[1, \infty)$ such that the improper integral

$$\int_1^\infty |f'(x)|\, dx$$

converges. Prove that the infinite series $\sum_1^\infty f(n)$ converges if and only if the integral $\int_1^\infty f(x)\, dx$ converges.

3. Show that

$$\int\int_{|z|<1} |1 + z^{2011}|^{2011} dxdy > \pi.$$

Here $z = x + iy$.

4. Let A and B be complex $n \times n$ matrices having the same rank. Suppose that $A^2 = A$ and $B^2 = B$. Prove that A and B are similar i.e there exists a nonsingular matrix P such that $A = P^{-1}BP$.

5. Find all pairs of integers a and b satisfying $0 < a < b$ and $a^b = b^a$.

6. Let $p \in \mathbb{N}$ be a prime. Prove: there is at most one pair of integers a, b with $0 < a \leq b$ such that $p = a^2 + b^2$.

7. Let $A = (a_{ij})$ be a real $n \times n$ matrix with nonnegative entries such that

$$\sum_{j=1}^{n} a_{ij} = 1 \quad (1 \leq i \leq n).$$

Prove that no eigenvalue of A has absolute value greater than 1.

8. Is there a polynomial f in $\mathbb{R}[X, Y]$ such that the image of f is $(0, \infty)$?