COLUMBIA – BARNARD

Mathematics Prize Exam

March 25th, 2010

This is a three-hour exam. It has eight questions. **Each question is worth 10 points**. Submit your paper even if you have done no more than one or two problems. For maximal credit give complete reasons for your answers. However, partial credit will be given for signicant progress made on a problem.

Please fill in the cover page and copy your exam number to each answer form. Do not staple together different problems.

1. Find the number of tilings of a $2 \times n$ rectangle with 1×2 and 2×1 dominos.

2. Let a, b, c be complex numbers on the unit circle with $a + b + c = 0$. Show that a, b, c form the vertices of an equilateral triangle.

3. Evaluate the limit

$$\lim_{n\to\infty} \cos\frac{\pi}{2^2}\cos\frac{\pi}{2^3}\cdots\cos\frac{\pi}{2^n}$$

4. Let A be an $n \times n$ real matrix, A^T its transpose. Show that $A^T A$ and A^T have the same range. In other words, given y, show that the equation $y = A^T Ax$ has a solution x if and only if the equation $y = A^T z$ has a solution z.

5. Let $f : [0, 1] \to \mathbb{R}$ be continuous and $f > 0$ on $[0, 1]$. Let $0 < x_1 < x_2 < \ldots < x_n = 1$ be a partition of $[0, 1]$ which divides the area under the graph of f into n equal parts, i.e

$$\int_0^{x_1} f(x)dx = \int_{x_1}^{x_2} f(x)dx = \ldots = \int_{x_{n-1}}^{x_n} f(x)dx.$$

Find

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \ldots + x_n}{n}.$$

6. Let a_1, $a_2, \ldots, a_{10}$ be integers with $1 \le a_i \le 25$, for $1 \le i \le 10$. Prove that there exist integers n_1, $n_2, \ldots, n_{10}$, not all zero, such that

$$\prod_{i=1}^{10} a_i^{n_i} = 1.$$

7. Let S_{23} be the group of permutations of $\{1, 2, 3, \ldots, 23\}$. Let G be an abelian subgroup of order 35 of S_{23}. Show that there exists $i \in \{1, 2, 3, \ldots, 23\}$ such that $\sigma(i) = i$ for all $\sigma \in G$.

8. Let f be a real valued continuous function on $[0, \infty)$ such that

$$\lim_{x\to\infty} \left(f(x) + \int_0^x f(t)\, dt \right)$$

exists. Show that $\lim_{x\to\infty} f(x) = 0$.