COLUMBIA – BARNARD

Mathematics Prize Exam

March 27th, 2014

This is a three-hour exam. It has eight questions. **Each question is worth 10 points**. Submit your paper even if you have done no more than one or two problems. For maximal credit give complete reasons for your answers. However, partial credit will be given for significant progress made on a problem.

Please fill in the cover page and copy your exam number to each answer form. Do not staple together different problems.

1. Let A be an orthogonal $2n \times 2n$ matrix of real numbers. Show that there exists a vector v such that $Av = v$.

2. For which real numbers c, there is a straight line that intersects the curve

$$y = x^4 + x^3 + cx^2 + x + 1$$

in 4 distinct points.

3. Assume that $A = (a_{ij})$, $B = (b_{ij})$ are $n \times m$ matrices such that

$$\sum_{i,j} b_{ij} x_i y_j = 0 \quad \text{whenever} \quad \sum_{i,j} a_{ij} x_i y_j = 0.$$

Show that B is a multiple of A.

4. Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(0) = 1, \quad |f(z)| \leq 2|z|^{3/2} - 1, \quad |z| \geq 2.$$

5. Let $f : \mathbb{R} \to (0, +\infty)$. Assume that $y = y(t)$ solves

$$y' = f(y + t \sin t).$$

Show that $y \to \infty$ as $t \to \infty$.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}$, be a function of 2 variables which is convex in each separate variable: $f(x, y)$ is convex in x for each fixed y and convex in y for each fixed x. Assume $f(A) = f(B) = f(C) = 0$ with $A = (1, 0)$, $B = (0, 1)$, $C = (-1, 1)$. Does it follow that $f(0, 0) \leq 0$? Prove or disprove.

7.You start with a bag with N balls in it, each with a unique color. You reach in and randomly (uniformly) choose one ball in your left hand and one ball in your right hand. You then paint the ball in the left hand to be the same color as the ball in the right hand and put both balls back in the bag. You repeat this procedure until all balls are the same color. As a function of N, what is the expected number of repetitions before all balls are the same color? For $N = 2$ this takes a single repetition – what about $N = 3$?

8. a) Prove that any group of order p^2, where p is a prime number, is abelian.
b)Prove that every group G of order 45 is abelian.