Mathematics Prize Exam

March 31st, 2022

This is a three-hour exam. It has eight questions. **Each question is worth 10 points**. Submit your paper even if you have done no more than one or two problems. For maximal credit give complete reasons for your answers. However, partial credit will be given for significant progress made on a problem.

1. Find all differential functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h},$$

for all $x \in \mathbb{R}$, and $h \neq 0$.

2. Let A be an $m \times n$ matrix of rank r and B a $p \times q$ matrix of rank s. Find the dimension of the vector space of $n \times p$ matrices X such that $AXB = 0$.

3. Show that $x^3 - 2x$ is an injective function from the rational numbers to the rational numbers.

4. Let P be a convex polygon in the plane. Show that there exists a pair of perpendicular lines which split P into 4 regions of equal area.

5. Let

$$p(z) = a_0 + a_1 z + .. + a_n z^n,$$

be a polynomial with real coefficients that satisfy $0 < a_0 \leq a_1 \leq .. \leq a_n$.

Show that all the roots of p lie in the unit disk $|z| \leq 1$.

6. In the Euclidean space $\mathbb{R}^4$, consider the "hyper-ellipsoid"

$$2x^2 + 3y^2 + 4z^2 + 5t^2 = 1.$$

Does there exist a 3-dimensional linear subspace passing through the origin which intersects the ellipsoid in a sphere?

7. Choose, at random, three points on the circle $x^2 + y^2 = 1$. Interpret them as cuts that divide the circle into three arcs. Compute the expected length of the arc that contains the point $(1, 0)$.

8. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function with $f(0) = 1$, and $f(f(x)) = x$. Evaluate

$$\int_0^1 (f(x) - x)^{2022} dx.$$