# Mathematics Prize Exam

## March 28th, 2019

This is a three-hour exam. It has eight questions. **Each question is worth 10 points**. Submit your paper even if you have done no more than one or two problems. For maximal credit give complete reasons for your answers. However, partial credit will be given for significant progress made on a problem.

Please fill in the cover page and copy your exam number to each answer form. Do not staple together different problems.

**1.** Let $A$ be a $2 \times 2$ real valued matrix such that
$$\det(A + A^T) = 20, \quad \det(2A + A^T) = 19.$$
Find $\det A$.

**2.** Assume that $f : \mathbb{R} \to \mathbb{R}$ has continous second derivatives and
$$|f''(x)| \le f(x), \qquad f(0) = 0.$$
Show that $f(x) = 0$ for all $x$.

**3.** Two tokens start at the origin in the plane and in the $k$th step ($k = 1, 2..$) each of them is moved a distance $a^k$ to either north, east, south or west, where $a$ is a positive rational number. After a number of steps the two tokens end up in the same spot but have not taken exactly the same path. Find all possible values of $a$.

**4.** Show that the two curves
$$2x = (y + 2)\cos(xy + 2x), \qquad 2y = x\cos(xy + 2x).$$
intersect at some point in the unit disk $\{x^2 + y^2 < 1\}$.

**5.** Let $d(x)$ denote the greatest common divisor of the entries of a vector $x \in \mathbb{Z}^n$. Show that if $A \in \mathbb{Z}^{n\times n}$ is a matrix then
$$|\det A| = 1 \text{ is equivalent to } d(x) = d(Ax) \text{ for all } x \in \mathbb{Z}^n.$$

**6.** Start with $n$ strings, which of course have $2n$ ends. Then randomly pair the $2n$ ends and tie together each pair. (Therefore you join each of the $n$ randomly chosen pairs.) Let $L$ be the number of resulting loops. Compute $E(L)$ the expected value for $L$.

**7.** Show that if $f, g$ are holomorphic functions such that $|f(z)|+|g(z)|$ is constant in the unit disk then $f$ and $g$ must be constant.

**8.** Let $f : [a, b] \to [a, b]$ be such that $f' \ge 0$. Show that there exists $c \in (a, b)$ such that
$$f(f(b)) - f(f(a)) = (f'(c))^2(b - a).$$