Mathematics Prize Exam

March 27th, 2018

This is a three-hour exam. It has eight questions. **Each question is worth 10 points**. Submit your paper even if you have done no more than one or two problems. For maximal credit give complete reasons for your answers. However, partial credit will be given for significant progress made on a problem.

Please fill in the cover page and copy your exam number to each answer form. Do not staple together different problems.

1. Find the surface area of the solid obtained by intersecting the cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$.

2. Let T_i be a finite collection of triangles in the plane. Show that if any 3 triangles from the collection have a nonempty intersection then they all have a point in common.

3. Show that the volume of the solid

$$E = \{x, y, z \geq 0 \quad | \quad xyz \geq 1 \geq ax + by + cz\}$$

for some positive constants a, b, c depends only on the value of $a \cdot b \cdot c$.

4. Find the largest possible value of the sum

$$\sum_{i=1}^{n} a_i \cos(a_1 + a_2 + ... + a_i),$$

with $n \geq 1$ arbitrary and $a_i \geq 0$ such that $\sum_i^n a_i = \pi$.

5. Let A, B be 2×2 real valued matrices such that $(AB)^2 = 0$. Prove or disprove that we must also have $(BA)^2 = 0$.

6. Let $u(x)$ be a solution to the second order ODE

$$u'' + x^2 u = 0.$$

Show that there exists an increasing sequence $x_k \to \infty$ such that

$$u(x_k) = 0, \quad \text{and} \quad x_{k+1} - x_k \to 0 \quad \text{as } k \to \infty.$$

7. Show that there exists a constant C such that line integral

$$\int_{S_1} \sin(nx)\, ds \leq C n^{-1/2},$$

where S_1 is the unit circle $x^2 + y^2 = 1$.

8. Find all holomorphic functions f defined in the unit disk $D = \{|z| < 1\}$ such that $f(\frac{1}{n} + ie^{-n})$ is real for all integers $n \geq 2$.