COLUMBIA – BARNARD

Mathematics Prize Exam

March 28th, 2017

This is a three-hour exam. It has eight questions. **Each question is worth 10 points**. Submit your paper even if you have done no more than one or two problems. For maximal credit give complete reasons for your answers. However, partial credit will be given for significant progress made on a problem.

Please fill in the cover page and copy your exam number to each answer form. Do not staple together different problems.

1. A king travels from the bottom-left corner of a chess board to the top-right corner, while visiting each of the 64 squares exactly once. Show that the king must have moved diagonally at least once.

2. Let $\mathbf{Z}^+$ denote the nonnegative integers. Suppose that $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ is a function satisfying
1) $f(3n) = 2f(n)$,
2) $f(3n+1) = f(3n) + 1$,
3) $f(3n+2) = f(3n) + 2$ for all $n \in \mathbf{Z}^+$.
Find, with proof, the number of values $0 \leq n \leq 2017$ such that $f(2n) = 2f(n)$.

3. Is there a converging series $\sum \frac{1}{a_k}$ of positive reals a_k that satisfy the following inequalities for all positive integers n:

$$a_1 + a_2 + \ldots + a_n \leq n^2 \qquad ?$$

4. Let a_i, $b_i \neq 0$, with $i \in \{1, \cdots, n\}$ be real numbers such that

$$\sum_{i=1}^{n} a_i |\sin(b_i x + i)| \geq 0, \qquad \forall x \in \mathbb{R}.$$

Show that $\sum_i a_i \geq 0$.

5. Let A and B be real $n \times n$ matrices such that $A^3 = B^5 = I$ and $AB = BA$. Show that $A + B$ is invertible.

6. Let $a, b \in \mathbb{C}$ and let $f : \mathbb{C} \to \mathbb{C}$ be a nonconstant holomorphic function such that $f(az + b) = f(z)$ for all $z \in \mathbb{C}$. Show that $a^n = 1$ for some integer $n \geq 1$.

7. Let f be a continuous function on $[0,1]$. What is

$$\lim_{n\to\infty} \int_{[0,1]^n} f\left((x_1 \cdots x_n)^{1/n}\right) \, dx_1 \cdots dx_n \qquad ?$$

8. Let $f(x,y)$ be a $C^\infty(\mathbb{R}^2)$ function of two variables such $f_{xx} \cdot f_{yy} = 0$.
Show that either f_{xx} or f_{yy} vanish identically in $\mathbb{R}^2$.
(Here f_{xx}, f_{yy} denote second order partial derivatives of f.)