COLUMBIA – BARNARD

Mathematics Prize Exam

March 29th, 2016

This is a three-hour exam. It has eight questions. **Each question is worth 10 points**. Submit your paper even if you have done no more than one or two problems. For maximal credit give complete reasons for your answers. However, partial credit will be given for significant progress made on a problem.

Please fill in the cover page and copy your exam number to each answer form. Do not staple together different problems.

1. What is the largest number of 1's an invertible matrix of size n with entries in $\{0,1\}$ can have? You must show both that this number is possible and that no larger number is possible.

2. For each x compute the limit

$$\lim_{n\to\infty} n\left(\left(1+\frac{x}{n}\right)^n - e^x\right).$$

3. A strip of width w is the set of all points which lie on, or between, two parallel lines distance w apart. Let S be a set of n ($n \geq 3$) points on the plane such that any three different points of S can be covered by a strip of width 1. Prove that S can be covered by a strip of width 2.

4. Let $a \neq 0$ and $\lambda \in \mathbb{C}$. Show that the matrix

$$M = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & \lambda \end{pmatrix}$$

is not diagonizable. (This means that there is no invertible matrix S such that $S^{-1}MS$ is diagonal.)

5. Let $z_1, .., z_n$ be complex numbers. Show that there exists a subset $J \subset \{1, .., n\}$ such that

$$\left|\sum_{j\in J} z_i\right| \geq \frac{1}{4\sqrt{2}} \sum_{j=1}^{n} |z_j|.$$

6. If $a_n > 0$ and $\sum \frac{1}{a_n}$ converges then also $\sum \frac{1}{b_n}$ converges where

$$b_n := \frac{a_1 + ... + a_n}{n}.$$

7. Let $a(x)$ be a positive function and assume that $y(x)$ satisfies in $[0,1]$ the ODE

$$y''(x) + \lambda a(x) y(x) = 0,$$
$$y(0) = 0, \quad y'(1) = 0.$$

for some constant λ. If y is not identically 0 show that $\lambda > 0$.

8. a) Consider a finite list of primes $p_1 < p_2 < \ < p_k$ in $\mathbb{N}$. Given integers m, n in $\mathbb{N}$, what is the probability that none of the p_i divide $gcd(m,n)$?

b) What is the probability that integers m and n have no common odd factors? (You may use that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.)