COLUMBIA – BARNARD

Mathematics Prize Exam

March 31th, 2015

This is a three-hour exam. It has eight questions. **Each question is worth 10 points**. Submit your paper even if you have done no more than one or two problems. For maximal credit give complete reasons for your answers. However, partial credit will be given for significant progress made on a problem.

Please fill in the cover page and copy your exam number to each answer form. Do not staple together different problems.

1. A computer is programmed to randomly generate a string of six symbols using only the letters A, B , C. What is the probability that the string will not contain three consecutive A's?

2. Evaluate

$$\int\int_D \frac{x^3}{x^2+y^2} dA$$

where D is the region

$$D := \{(x-1)^2 + y^2 \leq 1\} \cap \{y \geq 0\}.$$

3. Let $\ell_1, \ldots, \ell_{n+1}$ be lines through the origin in $\mathbf{R}^n$, no n of which are contained in a proper linear subspace. Show that there is a linear transformation A which, for $i \leq n$, takes ℓ_i to a line through the ith standard basis vector e_i, but which also takes ℓ_{n+1} to a line through $e_1 + \cdots + e_n$.

4. The number

$$2198214591730833304870133 69$$

is the thirteenth power of a positive integer. Which positive integer?

5. Let $f : [0,1] \to [0,1]$ be continuous such that $f(f(f(x))) = x$.
Show that $f(x) = x$.

6. If $a_n > 0$ and $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges then also $\sum_{n=1}^{\infty} \frac{1}{b_n}$ converges where

$$b_n := \frac{a_1 + \ldots + a_n}{n}.$$

7. Let A and B be complex $n \times n$ matrices such that $AB = BA^2$, and assume A has no eigenvalues of absolute value 1. Prove that A and B have a common (nonzero) eigenvector.

8. Show that the solution to the ODE:

$$y'' - (y')^2 + y = 0 \quad y(0) = a > 0, \quad y'(0) = 0,$$

is periodic.