Midterm 2

Name:

Problem	1	2	3	4	**Total:**
Max	25	25	25	25	100
Scores					

Instructions

- Time for test: **75 minutes**.
- Write clearly in pen or pencil, and make your final answer easy to find.
- To receive any partial credit you must **clearly** show your working.
- Do not use any notes or any textbooks.
- Do not communicate with any other student for any reason during the test.

1. Consider the problem

$$u_{tt} = u_{xx}, \quad 0 < x < \pi,$$

with the boundary conditions (BC)

$$u_x(0,t) = 1, \quad u(\pi,t) = \pi,$$

and the initial conditions (IC)

$$u(x,0) = x + 4\cos\frac{5}{2}x, \quad u_t(x,0) = 0.$$

(a) Set $v = u - x$. What problem does v solve?

(b) Find the eigenvalues λ_n and eigenvectors X_n associated to the equation/BC satisfied by v. You may assume that the eigenvalues must be positive.

(c) Write a solution v to the boundary value problem in series form.

(d) Find the Fourier coefficients in the system of eigenvectors X_n, for the initial conditions satisfied by v.

(e) Write the solution u to the given problem.

Solution. (a) v solves the problem

$$v_{tt} = v_{xx}, \quad 0 < x < \pi,$$

with the boundary conditions (BC)

$$v_x(0,t) = 0, \quad v(\pi,t) = 0,$$

and the initial conditions (IC)

$$v(x,0) = 4\cos\frac{5}{2}x, \quad v_t(x,0) = 0.$$

(b) Find separated solutions in the form $X(x)T(t)$. Then,

$$XT'' = X''T \Rightarrow \frac{T''}{T} = \frac{X''}{X} = -\lambda.$$

The associated eigenvalues problems are:

$$X'' + \lambda X = 0 \quad T'' + \lambda T = 0.$$

At $x = 0$ the solution XT must satisfy $X'(0)T(t) = 0 \Rightarrow X'(0) = 0$.
At $x = \pi$ the solution XT must satisfy $X(\pi)T(t) = 0 \Rightarrow X(\pi) = 0$.
Hence we must solve,

$$X'' + \lambda X = 0 \quad X'(0) = X(\pi) = 0.$$

Set $\lambda = \beta^2$, then

$$X = C\cos\beta x + D\sin\beta x.$$

$$X'(0) = 0 \Rightarrow D = 0.$$

Thus,

$$X = C\cos\beta x.$$

$X(\pi) = 0 \Rightarrow \cos\beta\pi = 0 \Rightarrow \beta = (n + 1/2), \quad n = 0, 1, ...$
Hence
$\lambda_n = (n + 1/2)^2, \quad X_n = \cos(n + 1/2)x.$
From $T' + \lambda_n T = 0$ we then get

$$T_n = A_n\cos(n + 1/2)t + B_n\sin(n + 1/2)t.$$

(c) From part (b) the solution in series form is given by

$$v = \sum_{n=0}^{\infty}(A_n\cos(n + 1/2)t + B_n\sin(n + 1/2)t)\cos(n + 1/2)x.$$

(d) By inspection,

$$v(x, 0) = 4\cos\frac{5}{2}x = \sum_{n=0}^{\infty} A_n\cos(n+1/2)x \Rightarrow A_2 = 4, A_n = 0 \quad n \neq 2.$$

$$v_t(x,0) = 0 = \sum_{n=0}^{\infty}(n+1/2)B_n \cos(n+1/2)x \Rightarrow B_n = 0.$$

(e) From part (c) and part (d)

$$u = v + x = 4\cos\frac{5}{2}t\cos\frac{5}{2}x + x.$$

2. Consider the sequence of eigenfunctions X_n from problem 1.
   (a) Use Green's second identity to prove that $(X_n, X_m) = 0$, for $m \neq n$.
   (b) Find the series expansion for $f(x) = 1$ in the system of eigenvectors X_n.
   (c) Does it converge in the uniform sense?
   (d) Does it converge in the L^2 sense?
   (e) Use Parseval's identity to find the sum of the numeric series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Solution.

(a)

$$(\lambda_n - \lambda_m)(X_n, X_m) = \int_0^\pi (-X_n'' X_m + X_n X_m'')dx = (-X_n' X_m + X_n X_m')|_0^\pi = 0$$

because $X_n'(0) = X_m'(0) = X_n(\pi) = X_m(\pi) = 0$.

(b)

$$A_n = \frac{(f, X_n)}{\|X_n\|^2}.$$

Compute

$$(f, X_n) = \int_0^\pi X_n dx = \int_0^\pi \cos(n+1/2)x dx = \sin(n+1/2)\pi/(n+1/2) = 2(-1)^n/(2n+1).$$

Compute

$$\|X_n\|^2 = \int_0^\pi X_n^2 dx = \int_0^\pi \cos^2(n+1/2)x dx = \int_0^\pi (1+\cos 2(n+1/2)x)/2 dx = \pi/2.$$

Thus,

$$A_n = \frac{4}{\pi}\frac{(-1)^n}{2n+1}.$$

(c) No, because f does not satisfy the same BC as the X_n's.

(d) Yes, because $\|f\| < \infty$.

(e)By Parseval,

$$\sum_0^\infty A_n^2 \|X_n\|^2 = \int_0^\pi f^2 dx.$$

Thus

$$\sum_0^\infty \frac{4}{(2n+1)^2} = \pi \Rightarrow \sum_0^\infty \frac{1}{(2n+1)^2} = \pi/4.$$

3. Solve $u_{tt} = 4u_{xx}$ in $0 < x < \infty, u(x,0) = u_t(x,0) = x$,

$$u(0,t) + u_x(0,t) = 0.$$

   (a) Introduce an appropriate function v which satisfies a wave equation with homogeneous Dirichlet boundary condition at zero, and apply the reflection method to find v.

   Important. You are allowed to use the formula for the solution for the wave equation on the whole line. However, you must exhibit all the steps yielding to the formula for v.

   (b) Find u by solving a first order PDE. Notice that you can easily solve this PDE by guessing that $u(x,t) = g(x)f(t)$ and using the IC/BC which must be satisfied by u. Be careful, you should have two different answers in the two regions $x > ct$ and $x < ct$ respectively.

Solution.

(a) Set $v = u + u_x$. Then v solves $v_{tt} = 4v_{xx}$ in $0 < x < \infty$, with

$$v(x,0) = v_t(x,0) = x + 1, \quad v(0,t) = 0.$$

Let $\phi = x + 1$ and ϕ_{odd} be the odd reflection of ϕ. Then according to formulas (2) and (3) on pages 59/60 in the textbook, our solution v is given by:

$$v(x,t) = x + 1 + \frac{1}{4}\int_{x-2t}^{x+2t} (y+1)dy, \quad x > 2t$$

$$v(x,t) = x + \frac{1}{4}\int_{2t-x}^{2t+x} (y+1)dy, \quad 0 < x < 2t.$$

Thus,

$$v(x,t) = (x+1)(t+1), \quad x > 2t$$

$$v(x,t) = x(t + 3/2), \quad 0 < x < 2t.$$

(b) Solve $v = u + u_x$ with $x > 2t$ and BC $u(x,0) = u_t(x,0) = x$.
Look for $u = g(x)f(t)$, then

$$gf + g'f = (x+1)(t+1) \Rightarrow f = t+1, g = x$$

and the BC is also satisfied.
Solve $v = u + u_x$ with $0 < x < 2t$ and BC $u(0,t) + u_x(0,t) = 0$.
Look for $u = g(x)f(t)$, then

$$gf + g'f = x(t+3/2) \Rightarrow f = t+3/2, g = x-1$$

and the BC is also satisfied.

4. Use the subtraction method to solve the heat equation

$$u_t = u_{xx}, \quad 0 < x < \infty,$$

with IC/BC

$$u(x,0) = \phi(x), u_x(0,t) = h(t).$$

Important. You are allowed to use the formula for the solution for the heat equation with a source on the whole line. However, you must exhibit all the steps yielding to the formula for the solution of the heat equation on the half-line, with homogeneous Neumann boundary condition.

Solution. Set

$$v = u - hx.$$

Then v solves,

$$v_t = v_{xx} - h'x, \quad 0 < x < \infty,$$

with IC/BC

$$u(x,0) = \phi(x) - h(0)x, u_x(0,t) = 0.$$

Set

$$f(x,t) = -h'(t)x, \quad \tilde{\phi} = \phi(x) - h(0)x.$$

Let $f_{ev}, \tilde{\phi}_{ev}$ be the even reflections (in the x variable) of f and $\tilde{\phi}$ across zero. Then according to formula (2) page 65 in the textbook, our solution is the restriction to $x > 0$ of the function:

$$v(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\tilde{\phi}_{ev}(y)dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)f_{ev}(y,s)dyds.$$

Thus,

$$v(x,t) = \int_0^{+\infty} S(x-y,t)\tilde{\phi}(y)dy + \int_{-\infty}^0 S(x-y,t)\tilde{\phi}(-y)dy$$

$$+ \int_0^t \int_0^{+\infty} S(x-y,t-s)f(y,s)dyds + \int_0^t \int_{-\infty}^0 S(x-y,t-s)f(-y,s)dyds.$$

Changing $-y$ into y in the second and fourth integrals we get,

$$v(x,t) = \int_0^{+\infty} [S(x-y,t) + S(x+y,t)]\tilde{\phi}(y)dy$$

$$+ \int_0^t \int_0^{+\infty} [S(x-y,t-s) + S(x+y,t-s)]f(y,s)dyds.$$