Midterm I: Partial Differential Equations

Name:

Problem	1	2	3	4	5	**Total:**
Max	15	20	20	25	20	100
Scores						

TOTAL: /100

Instructions

- Time for test: **75 minutes**.
- Write clearly in pen or pencil, and make your final answer easy to find.
- To receive any partial credit you must **clearly** show your working.
- Do not use any notes or any textbooks.
- Write first and last name.
- Do not communicate with any other student for any reason during the test.

Problem 1. Let $u_1(x,t), u_2(x,t)$ be two solutions to the heat equation $(k > 0,)$

$$u_t = ku_{xx}, \quad 0 < x < l, t > 0,$$

with initial and boundary conditions

$$u_i(x,0) = g_i(x), u_i(0,t) = f_i(t), u_i(l,t) = h_i(t), \quad i = 1,2.$$

Assume that $g_1 \leq g_2, f_1 \leq f_2$ and $h_1 \leq h_2$. Show that $u_1 \leq u_2$ in the set $R := [0,l] \times [0,\infty)$.

Hint: Set $w = u_1 - u_2$ and show that $w \leq 0$ in R.

Solution. Set $w = u_1 = u_2$. By the linearity of the heat equation, w solves

$$w_t = kw_{xx}, \quad 0 < x < l, t > 0,$$

with initial and boundary conditions

$$w(x,0) = g_1 - g_2, w(0,t) = f_1 - f_2, w(l,t) = h_1 - h_2.$$

Thus, according to our assumption

$$w(x,0) \leq 0, w(0,t) \leq 0, w(l,t) \leq 0,$$

for all t's. By the maximum principle,

$$w(x,t) \leq \max_{t=0,x=0,x=1} w \leq 0,$$

on R. Thus $u_1 \leq u_2$ on R.

Problem 2. Consider the following problem:

$$\begin{cases} u_t = u_{xx} + 2(t+1) + x(1-x) \\ u(0,t) = 0, u(1,t) = 0 \\ u(x,0) = x(1-x) \end{cases} \tag{1}$$

for $0 < x < 1$ and $t > 0$.

(1) Verify that $u(x,t) = (t+1)x(1-x)$ is a solution to (1).
(2) Find the maximum M and the minimum m of the initial and boundary data.
(3) Show that for all $t > 0$ the function $u(x,t)$ exceeds M at a certain point inside $[0,1]$.
(4) Does the maximum principle hold for this problem?

Solution.
(1) Compute

$$u_t = x(1-x), u_{xx} = -2(t+1).$$

Clearly,

$$\begin{cases} u_t = u_{xx} + 2(t+1) + x(1-x) \\ u(0,t) = 0, u(1,t) = 0 \\ u(x,0) = x(1-x) \end{cases}$$

(2) Clearly the minimum m of the boundary data is zero. The maximum M is $1/4$ and it is achieved at $x = 1/2$ by the initial condition $x(1-x)$.

(3) Since $u(x,t) = (t+1)x(1-x)$, then

$$u(1/2,t) = (t+1)/4 > M,$$

at any time $t > 0$.

(4) No. Part (3) violates the maximum principle.

Problem 3. Consider the inhomogeneous problem

$$\begin{cases} u_t - ku_{xx} = f(x,t) \\ u(0,t) = g(t), u(l,t) = h(t) \\ u(x,0) = \phi(x) \end{cases} \tag{2}$$

where $k > 0$, $0 < x < l$ and $t > 0$.

(1) Are the boundary conditions for this problem of Dirichlet, Neumann or Robin type?
(2) Prove the uniqueness of a solution for (2) using the energy method.(No credit will be given if you use the maximum principle.)

Solution.
(1) Dirichlet.

(2) This is showed on page 42 in our textbook.

Problem 4. Consider the initial value problem

$$\begin{cases} 3u_{tt} + u_{xx} - 4u_{xt} = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases} \tag{3}$$

for $-\infty < x < \infty$ and $t > 0$.

(1) Of what type (parabolic, hyperbolic, elliptic) is the equation in (3)?

(2) Write the solution for (3).

Hint: Factor the operator as we did for the wave equation. Do not guess a solution. Show all your work to obtain the general solution as we did for the wave equation.

Solution.

(a) $a_{11} = 1, a_{22} = 3, a_{12} = -2 \Rightarrow a_{12}^2 - a_{11}a_{22} > 0 \Rightarrow$ *Hyperbolic*

(b) Factor the operator as

$$(\partial_t - \partial_x)(\partial_t - \frac{1}{3}\partial_x)u = 0.$$

Set

$$v = u_t - \frac{1}{3}u_x, \quad v_t - v_x = 0.$$

Then

$$v = h(x+t)$$

and

$$(1) \quad u_t - \frac{1}{3}u_x = h(x+t).$$

(1) has the homogeneous solution $u_h = g(x+1/3t)$ and the particular solution $u_p = f(x+t)$ with $f'(s) = 3/2h(s)$. Since h is arbitrary we have

$$u = f(x+t) + g(x+1/3t)$$

with f, g arbitrary.

Clearly if we choose $f = g = 0$ the initial conditions are satisfied.

Problem 5. Use the coordinate method to solve

$$u_x + u_y = u^2.$$

Solution. Set

$$x' = x + y, y' = x - y.$$

Then the equation becomes

$$u_{x'} = \frac{u^2}{2}.$$

Thus,

$$\frac{u_{x'}}{u^2} = \frac{1}{2} \Leftrightarrow (-1/u)_{x'} = 1/2.$$

Integrating

$$-1/u = 1/2x' + C(y') \Leftrightarrow u = -2/(x' + 2C(y')) \Leftrightarrow u = -2/(x + y + 2C(x - y))$$

for any arbitrary function C.