HOMEWORK 2 SOLUTIONS

SECTION 1.5 : $\#1, 2, 3, 4(a) - (b)$

- #1) Solve the boundary value problem:

$$u_{xx} + u = 0$$

$$u(0) = 0 \text{ and } u(L) = 0.$$

Answer: The solution to the ODE is:

$$u(x) = A\sin(Bx + C)$$

If $A = 0$, then the solution is $u(x) \equiv 0$.
If $A \neq 0$, then we can solve for the variables by substituting the general solution into the ODE.

$$\begin{aligned} u_{xx} + u = (A - AB^2)\sin(Bx + C) &= 0 \\ A(1 - B^2) &= 0 \\ B &= \pm 1 \end{aligned}$$

Using the boundary conditions we have:

$$u(0) = A\sin(C) = 0$$

Then $C = n\pi$ with n an integer. Also,

$$\begin{aligned} u(L) = A\sin(BL + C) &= 0 \\ BL + C &= m\pi \\ \pm L + n\pi &= m\pi \\ L &= m\pi \end{aligned}$$

So, if $L = m\pi$ for some integer m, then $u(x) = A\sin(x + n\pi)$ is a solution.
If $L \neq m\pi$, then $u(x) \equiv 0$ is the only solution.

- #2) Consider the problem

$$u''(x) + u'(x) = f(x)$$

$$u'(0) = u(0) = \frac{1}{2}[u'(l) + u(l)]$$

(a) Does the following problem have a unique solution?
(b) Does a solution necessarily exist?

Answer: (a) The problem does not have a unique solution.

Proof. Let u_1 and u_2 be solutions to the problem. Let $v(x) = u_1(x) - u_2(x)$. Then $v(x)$ is a solution of

$$u''(x) + u'(x) = 0$$

$$u'(0) = u(0) = \frac{1}{2}[u'(l) + u(l)]$$

Then $v(x) = A + Be^{-x}$, and

$$\begin{aligned} v'(0) = v(0) &= \frac{1}{2}[v'(l) + v(l)] \\ -B = A + B &= \frac{1}{2}[-Be^{-l} + A + Be^{-l}] \\ -B = A + B &= \frac{A}{2} \\ A &= -2B \\ v(x) &= B(e^{-x} - 2), \text{ for any real number } B \end{aligned}$$

Then if $u(x)$ is a solution of the initial problem, $u_2(x) = u_1(x) + e^{-x} - 2$ is also a solution. □

(b)A solution does not necessarily exist unless $f(x)$ satisfies the condition

$$\int_0^l f(x)dx = 0$$

Proof. We integrate both sides of the ODE and use the conditions on u at $x = 0$ and at $x = l$.

$$\begin{aligned} \int_0^l [u''(x) + u'(x)]dx &= \int_0^l f(x)dx \\ u'(l) + u(l) - u'(0) - u(0) &= \int_0^l f(x)dx \\ 2u(0) - u(0) - u(0) &= \int_0^l f(x)dx \\ 0 &= \int_0^l f(x)dx \end{aligned}$$

□

- #3 Solve the boundary value problem

$$\begin{aligned} u''(x) = 0 \text{ if } 0 < x < 1, \\ u'(0) + ku(0) = 0 \\ u'(1) \pm ku(1) = 0 \end{aligned}$$

Answer: The solution to the ODE on the interval $0 < x < 1$ is:

$$u(x) = Ax + B$$

Using the boundary conditions, we have:

$$\begin{aligned} A &= -Bk \\ -Bk \pm k(-Bk + B) &= 0 \end{aligned}$$

Case 1 $(+k)$: The equations from the boundary conditions are

$$\begin{aligned} A &= -Bk \\ -Bk^2 &= 0 \end{aligned}$$

Then either $B = 0$ or $k = 0$, which means that $A = 0$. Then $u(x) = B$ with $B = 0$ if $k \neq 0$.

Case 2 $(-k)$: The equations from the boundary conditions are

$$\begin{aligned} A &= -Bk \\ Bk(k-2) &= 0 \end{aligned}$$

Then either $B = 0$, $k = 0$, or $k = 2$. Again, if $B = 0$ or $k = 0$, then $A = 0$. If $k = 2$, then $A = -2B$

$$u(x) = \begin{cases} B & \text{if } k = 0 \text{ and } B \in \mathbb{R}, \\ 0 & \text{if } k \neq 0 \text{ and } k \neq 2, \\ -2Bx + B & \text{if } k = 2\ . \end{cases}$$

The only degree 1 solution occurs when $k = 2$ in the "$-k$" case.

- #4 Consider the Neumann problem:

$$\begin{aligned} \Delta u &= f(x, y, z) \text{ in } D, \\ \partial_{\vec{n}} u &= 0 \text{ on boundary of } D. \end{aligned}$$

(a) What can we add to any solution to get another solution?

(b) Show that $\iiint_D f(x, y, z) dxdydz = 0$.

Answer: (a) We can add any solution to the homogeneous Neumann problem to any solution of the above Neumann problem. This include constants and $ax + by + cz$ with $\langle a, b, c\rangle \cdot \vec{n} = 0$.

(b) Using the divergence theorem and the information given in the PDE, we have:

$$\begin{aligned} \iiint_D f(x, y, z) dxdydz &= \iiint_D \Delta u dxdydz \\ &= \iiint_D \nabla \cdot \nabla u dxdydz \\ &= \iint_{\partial D} \vec{n} \cdot \nabla u dS \\ &= \iint_{\partial D} \partial_{\vec{n}} u dS \\ &= \iint_{\partial D} 0 dS = 0 \quad \square \end{aligned}$$

SECTION 1.6 : #1, 4

- #1 What are the types of the following equations?

  (a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$

  Answer: Remember that $u_{xy} = u_{yx}$, then the equation becomes:

$$u_{xx} - 4u_{xy} + u_{yy} + 2u_y + 4u = 0$$

$$\mathfrak{D} = \left(\frac{-4}{2}\right)^2 - (1)(1) = 3 > 0$$

  Then the equation is Hyperbolic.

  (b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$

  Answer: $\mathfrak{D} = \left(\frac{6}{2}\right)^2 - (9)(1) = 9 - 9 = 0$. Then the equation is Parabolic.

- #4 What is the *type* of the equation

$$u_{xx} - 4u_{xy} + 4u_{yy} = 0?$$

  Answer: $\mathfrak{D} = (-2)^2 - (1)(4) = 0$. It is Parabolic.

  Show by direct substitution that $u(x, y) = f(y + 2x) + xg(y + 2x)$ is a solution for arbitrary functions f and g. **Answer:** Compute the partial derivatives and then substitute.

$$\begin{aligned}
u_x &= 2f'(y+2x) + g(y+2x) + 2xg'(y+2x) \\
u_{xx} &= 4f''(y+2x) + 2g'(y+2x) + 2g'(y+2x) + 4xg''(y+2x) \\
&= 4f''(y+2x) + 4g'(y+2x) + 4xg''(y+2x) \\
u_{xy} &= 2f''(y+2x) + g'(y+2x) + 2xg''(y+2x) \\
u_y &= f'(y+2x) + xg'(y+2x) \\
u_{yy} &= f''(y+2x) + xg''(y+2x)
\end{aligned}$$

  Then substituting, we have:

$$\begin{aligned}
u_{xx} - 4u_{xy} + 4u_{yy} &= 4f''(y+2x) + 2g'(y+2x) + 2g'(y+2x) + 4xg''(y+2x) \\
&\quad - 8f''(y+2x) - 4g'(y+2x) - 8xg''(y+2x) \\
&\quad + 4f''(y+2x) + 4xg''(y+2x) \\
&= 0 \quad \square
\end{aligned}$$

SECTION 2.1 : #8, 9

- #8 Consider the spherical wave equation:

$$u_{tt} = c^2 \left(u_{rr} + \frac{2}{r} u_r \right)$$

(a) Change variables by making the substitution $v = ru$.

Answer: First solve for u, then take partial derivatives.

$$\begin{aligned} u &= \frac{1}{r} v \\ u_{tt} &= \frac{1}{r} v_{tt} \\ u_{rr} &= \frac{1}{r} v_{rr} - \frac{2}{r^2} v_r + \frac{2}{r^3} v \end{aligned}$$

Substituting into the spherical wave equation, we get

$$\begin{aligned} \frac{1}{r} v_{tt} &= \frac{1}{r} c^2 v_{rr} \\ v_{tt} &= c^2 v_{rr} \end{aligned}$$

(b) Solve for v using the fact that $u_{tt} = c^2 u_{xx}$ has a solution of the form

$$u(x,t) = f(x+ct) + g(x-ct)$$

Answer: All that has changed here are the variables, so

$$v(r,t) = f(r+ct) + g(r-ct)$$

is a solution of $v_{tt} = c^2 v_{rr}$, which means

$$u(r,t) = \frac{1}{r} f(r+ct) + \frac{1}{r} g(r-ct)$$

is a solution of the spherical wave equation.

(c) Solve the spherical wave equation with initial conditions

$$\begin{aligned} u(r,0) &= \phi(r) \\ u_t(r,0) &= \psi(r) \end{aligned}$$

with both ϕ and ψ even functions of r.

Answer: First, rewrite the initial conditions in terms of v, and then solve using the solution to the initial value problem wave equation given by (8) in the book.

$$\begin{aligned} v(r,0) &= r\phi(r) = L(r) \\ v_t(r,0) &= r\psi(r) = M(r) \end{aligned}$$

Then from the solution to the IVP, we have

$$v(r,t) = \frac{1}{2}[L(r+ct) + L(r-ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} M(s)ds$$
$$v(r,t) = \frac{1}{2}[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} s\psi(s)ds$$
$$u(r,t) = \frac{1}{2r}[(r+ct)\phi(r+ct) + (r-ct)\phi(r-ct)] + \frac{1}{2rc}\int_{r-ct}^{r+ct} s\psi(s)ds$$

- #9 Solve the initial value problem

$$\begin{aligned} u_{xx} - 3u_{xt} - 4u_{tt} &= 0, \\ u(x,0) &= x^2, \\ u_t(x,0) &= e^x. \end{aligned}$$

Answer: By factoring the operator acting on u, we have

$$-4(\partial_t - \frac{1}{4}\partial_x)(\partial_t + \partial_x)u = 0.$$

Noticing the similarities of this to the wave equation, which factors like

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0,$$

you may guess that the solution we're looking for has the form

$$u(x,t) = f(x-t) + g(x + \frac{1}{4}t).$$

One way to show this is to let $v = (\partial_x + \partial_t)u$. Then we have

$$\begin{aligned} u_t + u_x &= v \\ v_t - \frac{1}{4}v_x &= 0 \end{aligned}$$

Then $u_x + u_t = v(x,t) = h(x + \frac{1}{4}t)$.

$u_x + u_t = h(x + \frac{1}{4}t)$ has a homogeneous solution of the form $u_h = f(x-t)$. It also has a particular solution $u_p = g(x + \frac{1}{4}t)$ with $g'(s) = \frac{4}{5}h(s)$. Then

$$\begin{aligned} u(x,t) &= f(x-t) + g(x + \frac{1}{4}t) \\ u_t(x,t) &= -f'(x-t) + \frac{1}{4}g'(x + \frac{1}{4}t) \end{aligned}$$

Considering the initial conditions, we have

$$\begin{aligned} u(x,0) &= f(x) + g(x) = x^2 \\ u_t(x,0) &= -f'(x) + \frac{1}{4}g'(x) = e^x \end{aligned}$$

Solving for f and g, we integrate

$$\begin{aligned}\int e^x dx &= \int -f'(x) + \frac{1}{4}g'(x)dx \\ e^x &= -f(x) + \frac{1}{4}g(x) + c \\ e^x &= g(x) - x^2 + \frac{1}{4}g(x) + c\end{aligned}$$

So, we have

$$\begin{aligned}g(x) &= \frac{4}{5}\left(e^x + x^2\right) + c \\ f(x) &= \frac{-4}{5}e^x + \frac{1}{5}x^2 - c \\ u(x,t) &= \frac{4}{5}\left(e^{x+\frac{1}{4}t} - e^{x-t}\right) + x^2 + \frac{1}{4}t^2\end{aligned}$$